- discrete time systems:
  - time proceeds in clicks
  - "maps"
  - modeling tool: *difference* equation
- continuous time systems:
  - time proceeds smoothly
  - "flows"
  - modeling tool: *differential* equations

93

What do those beasts look like and how do we deal with them?

Difference equations:

- e.g., $x_{n+1} = cos(x_n)$
- given state x at time n, tells you state at time $n+1$
- solve by iterating

94

What do those beasts look like and how do we deal with them?

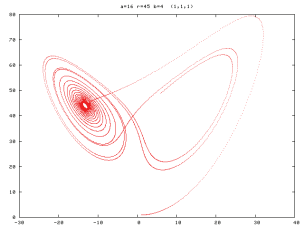
Difference equations:

- e.g., $x_{n+1} = R\, x_n\, (1 - x_n)$
- given state x at time n, tells you state at time $n+1$
- solve by iterating

Differential equations:

- e.g., Lorenz equations...

95



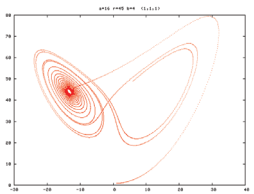
$x' = 16(y - x)$

$y' = 45x - y - xz$

$z' = xy - 4z$

96

Important note: in both maps and flows, *x* can be a vector!



$x' = 16(y - x)$

$y' = 45x - y - xz$

$z' = xy - 4z$

$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \rightarrow f \rightarrow \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix}$

State *vector*

97

What do those beasts look like and how do we deal with them?

Difference equations:

- e.g., $x_{n+1} = R\,x_n\,(1 - x_n)$
- given state x at time n, tells you state at time $n+1$
- solve by iterating

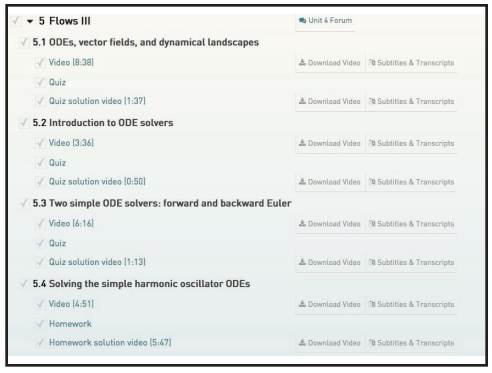
$x_n \rightarrow f \rightarrow x_{n+1}$

Differential equations:

- e.g., Lorenz equations
- given state x at time t, tells you *the direction in which that state will evolve*
- solve with an ODE solver (see Liz's notes, MOOC)

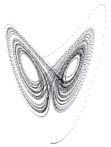
$x \rightarrow f \rightarrow x'$

98



99

The basic idea behind (one family of) ODE solvers:



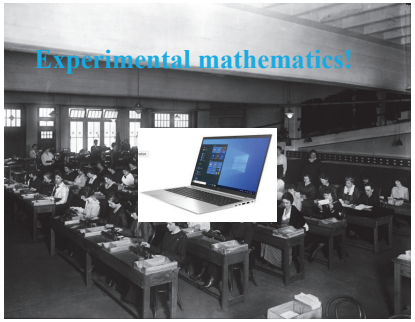
- Follow the slope that the ODE gives you
- Simplest: Euler
- More creative: legion…e.g., `ode45`, `ode34`

All very well if you have a nice modern computer…

100

www.computerhistory.org

101



www.computerhistory.org

102

CHAOS THEORY

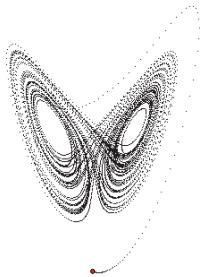
The Hidden Heroines of Chaos

10 |

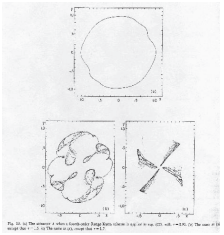
Two women programmers played a pivotal role in the birth of chaos theory. Their previously untold story illustrates the changing status of computation in science.

https://www.quantamagazine.org/hidden-heroines-of-chaos-ellen-fetter-and-margaret-hamilton-20190520/

103



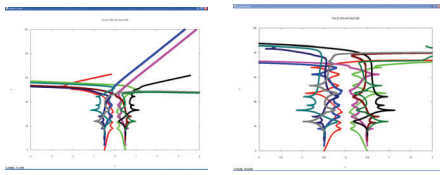
104



Different timestep

Lorenz, *Physica D* **35**:229

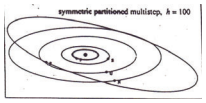
105



Different arithmetic

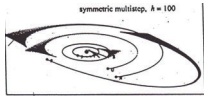
N. Ross Ph.D. thesis, Ucolorado, 2008

106

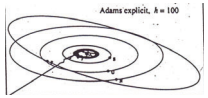


Different solver algorithm…

107



108



109

Moral: numerical methods can run amok in "interesting" ways…

- can cause distortions, bifurcations, etc.
- and these look a lot like *real, physical* dynamics…
- source: algorithms, arithmetic system, timestep, etc.
- Q: what could you do to diagnose whether your results included spurious numerical dynamics?

110

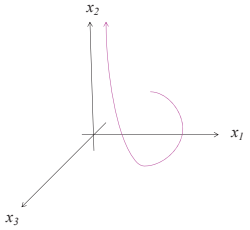
Moral: numerical methods can run amok in "interesting" ways…

- can cause distortions, bifurcations, etc.
- and these look a lot like *real, physical* dynamics…
- source: algorithms, arithmetic system, timestep, etc.
- Q: what could you do to diagnose whether your results included spurious numerical dynamics?
  - *change the timestep*
  - *change the method*
  - *change the arithmetic*

But beware machine ε…

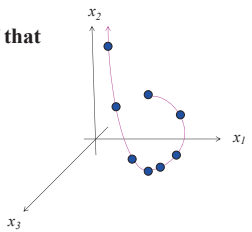
111

The state space representation suppresses time…



112

…but if you have evenly spaced samples of the trajectory, you can reconstruct some of that info:



(and that's what many ODE solvers give you)

113

But!

Many solvers, such as Matlab's `ode45`, are *adaptive*: they change the timestep and/or the method itself, on the fly, in order to correctly simulate the dynamics.

(The algorithms for this are interesting; we can talk about them offline.)

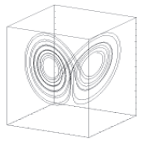
That means that the points that are output by tools like `ode45` are *not necessarily evenly spaced in time (*)*. That can matter, depending on how you're using that solution…

(*) unless you take steps to force that

114

So ODE solvers make mistakes.

…and chaotic systems are sensitively dependent on initial conditions….



…???

115

Shadowing lemma

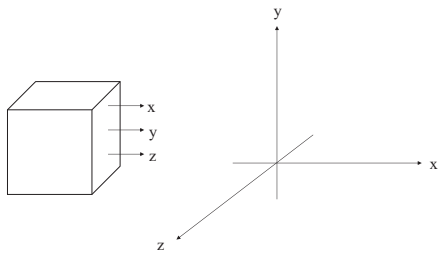
Every* noise-added trajectory on a chaotic attractor is *shadowed* by a true trajectory.

Important: this is for *state* noise, not *parameter* noise.

(*) Caveat: not if the noise bumps the trajectory out of the basin

116

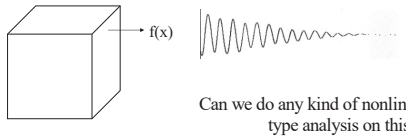
That state-space stuff is all very well, but it's a bit utopian.



117

Reality:

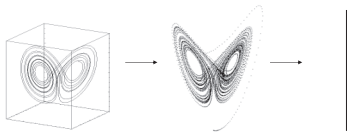
- Rarely do you even *know* what the state variables are.
- Even if you did, you might not be able to *measure* all of them.
- And even if you could, doing so might change the dynamics...
- But we can often measure at least *one* thing:



Can we do any kind of nonlinear-type analysis on this???

118

What we're really asking for when we do time-series analysis on scalar data…

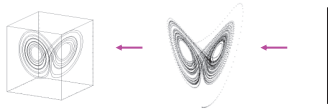


Undoing a projection =gack=

119

Delay-coordinate embedding

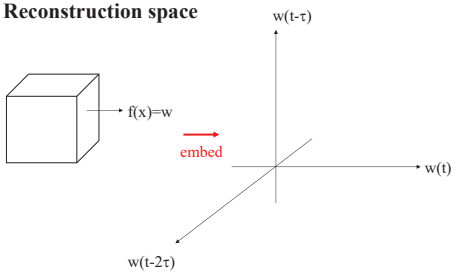
"reinflate" that squashed data to get a *topologically identical* copy of the original thing.



(not quite, but almost…)

120

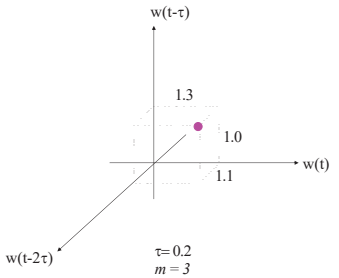
Reconstruction space



121

Mechanics

w	t
1.3	0.1
1.2	0.2
1.0	0.3
0.8	0.4
1.1	0.5
1.4	0.6
1.6	0.7



122

w	t
1.3	0.1
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0.8	0.4
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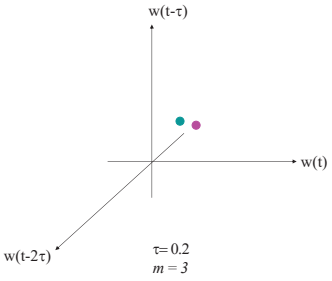
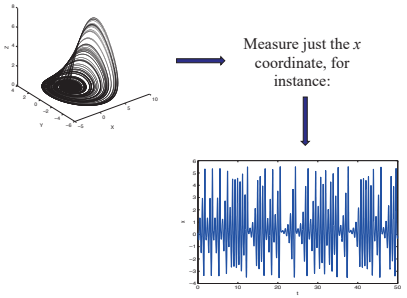
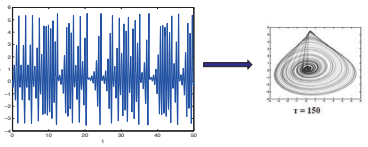


123



124

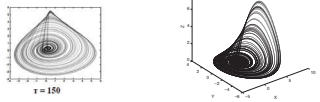
…and then embed:



125

Takens* theorem

For the right τ and enough dimensions, the embedded dynamics are diffeomorphic to (and thus have same topology as) the original state-space dynamics.



* Whitney, Mane, Crutchfield, …

126

Diffeomorphic: mapping from the one to the other is differentiable and has a differentiable inverse.

What that means:

- *qualitatively* the same shape (topology)

- i.e., can deform one into the other…

127

www.shapeways.com/shops/henryseg

128

Diffeomorphic: mapping from the one to the other is differentiable and has a differentiable inverse.

What that means:

- *qualitatively* the same shape (topology)

- i.e., can deform one into the other…

129

Diffeomorphic: mapping from the one to the other is differentiable and has a differentiable inverse.

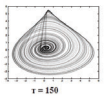
What that means:

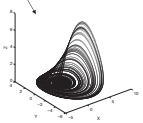
- *qualitatively* the same shape (topology)
- i.e., can deform one into the other
- have same dynamical invariants (e.g., λ)

130

Takens* theorem

For the right τ and enough dimensions, the embedded dynamics are diffeomorphic to (and thus have same topology as) the original state-space dynamics.



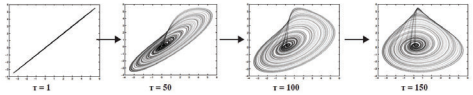


* Whitney, Mane, Crutchfield, …

Note: the measured quantity must be a smooth, generic function of at least one state variable, and must be uniformly sampled in time.

131

Choosing τ:



Fabulous talk about this matter:
www.youtube.com/embed/R-X0WXYp9vk

132

Choosing *m*

$m > 2d$: sufficient to ensure no crossings in reconstruction space (Takens et al.)…

…but that may be overkill, and you rarely know *d* anyway.

"Embedology" paper: $m > 2\, d_{box}$
(box-counting dimension)

133

Nonlinear time-series analysis (NLTSA):

- The bible: H. Kantz & T. Schreiber, *Nonlinear Time Series Analysis*
- Associated software: TISEAN
  www.mpipks-dresden.mpg.de/~tisean
- A short review article: EB & H. Kantz, "Nonlinear Time Series Analysis Revisited," *CHAOS* **25:**097610 (2015)

Units 8 & 9 in Liz's MOOC!

134

If Δt is not uniform

Theorem (Takens): for $\tau>0$ and $m > 2d$, reconstructed trajectory is diffeomorphic to the true trajectory

Conditions: evenly sampled in time, smooth generic measurement function

135

Interspike interval embedding

idea: lots of systems generate spikes — hearts, nerves, etc.

if you assume that the spikes are the result of an integrate-and-fire system, then the Δt has a one-to-one correspondence to some state variable's *integrated* value…

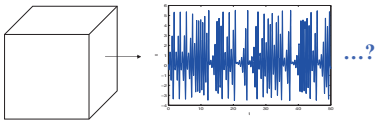
in which case the embedding theorems still hold.

(with the Δts as state variables)

Sauer *Chaos* **5**:127

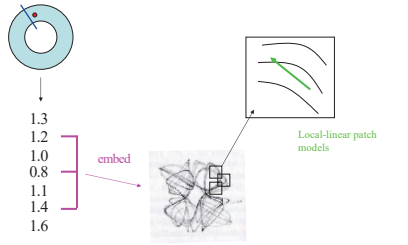
136

Prediction



137

Predicting the path of a roulette ball…



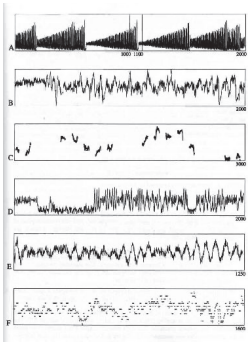
The Eudaemonic Pie
(or The Newtonian Casino)

138

The Santa Fe competition

- Weigend & Gershenfeld, 1992
- put a bunch of data sets up on an ftp server
- and invited all comers to predict their future
- chronicled in *Time Series Prediction: Forecasting the Future and Understanding the Past,* Santa Fe Institute, 1993 (from which the images on the following half-dozen slides were reproduced)

139



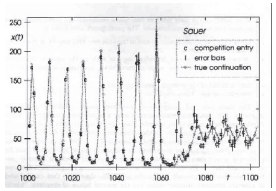
140

The Santa Fe competition: data

- Laboratory laser
- Medical data (sleep apnea)
- Currency rate exchange
- RK4 on some chaotic ODE
- Intensity of some star
- A Bach fugue

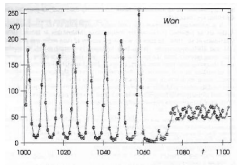
141

Embedding + patch models: (Sauer)



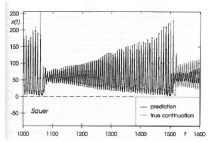
142

Neural net: (Wan)

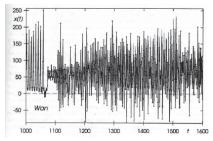


143

Further out:



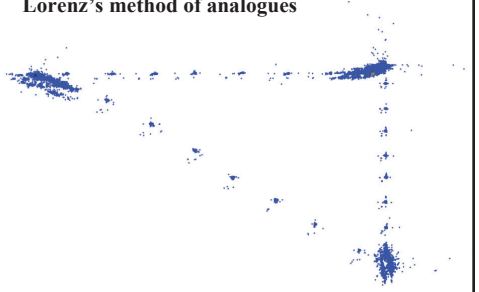
Sauer



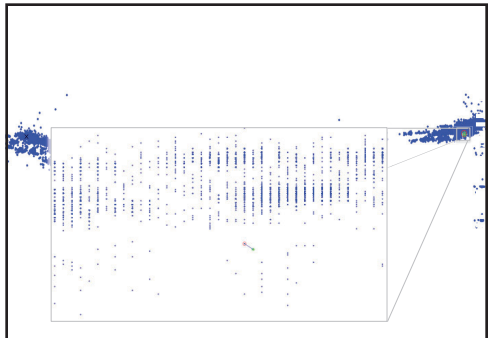
Wan

144

An even simpler prediction method:
Lorenz's method of analogues

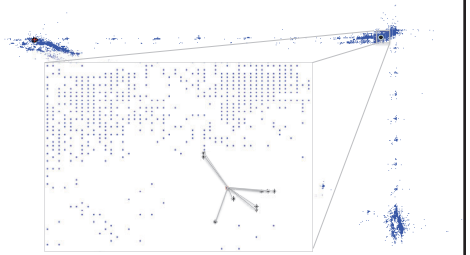


145

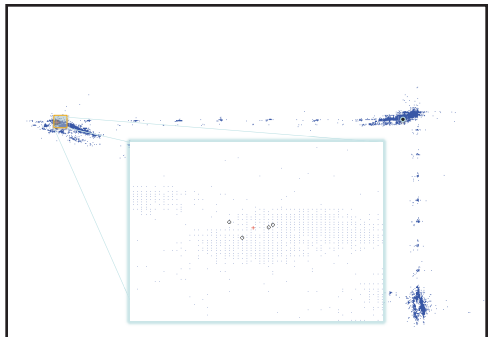


146

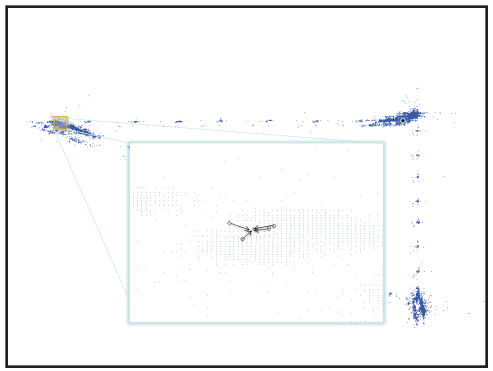
A *k*-nearest neighbor modification of LMA



147

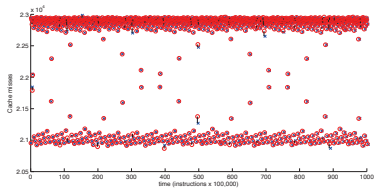


148



149

Using *k*LMA to predict computer dynamics



RMSPE = 11.566
(.07% of the average)

`column_major`
cache misses

Garland/Bradley *Intelligent Data Analysis* 2011

150

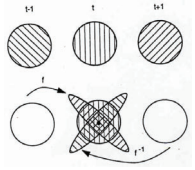
Noise…

Linear filtering: a bad idea if the system is chaotic

Nonlinear alternatives:

- use the stable and unstable manifold structure of the chaotic attractor…

151



Farmer & Sidorowich, in *Evolution, Learning and Cognition*, World Scientific, 1983

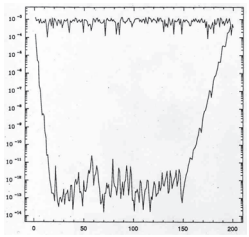
152

Idea:

- If you have a model of the system, you can simulate what happens to each point in forward *and backward* time
- If your system has transverse stable and unstable manifolds, that does useful things to the noise balls
- Since all three versions of that data should be identical at the middle time, can average them
- ⇒ noise reduction!
- Works best if manifolds are perpendicular, but requires only transversality

153

Results:



Farmer & Sidorowich, in *Evolution, Learning and Cognition*, World Scientific, 1983

154

Noise…

Linear filtering: a bad idea if the system is chaotic

Nonlinear alternatives:

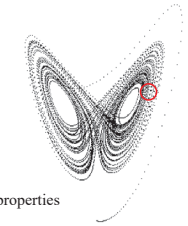
- use the stable and unstable manifold geometry of the chaotic attractor
- can also use the *topology* of the attractor

Topology-based signal separation

V. Robins
Department of Applied Mathematics, Research School of Physical Sciences and Engineering, The Australian National University, Canberra, ACT 0200 Australia
N. Rooney and E. Bradley
Department of Computer Science, University of Colorado, Boulder, Colorado 80309-0430

155

Chaos and control



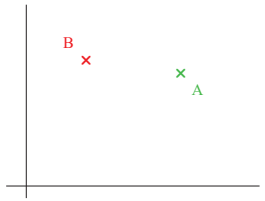
Can we use some of those cool properties and turn them to advantage?

e.g., dense attractor coverage? ◯

156

Control:

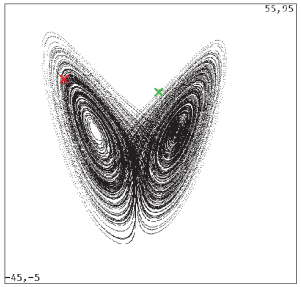
getting from A to B, minimizing some cost functional



157

Lorenz System:

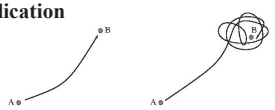
denseness, reachability, and control



R = 50

158

Denseness & reachability in a real engineering application



TRADITIONAL CONTROL

THIS IDEA

- can control position/volume/density of attractor — *within limits*
- possibly not reachable any other way
- not for time-critical applications (that "eventually")

Using Chaos to Broaden the Capture Range of a Phase-Locked Loop

Elizabeth Bradley, *Member, IEEE*

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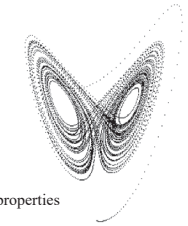
Chaos and control



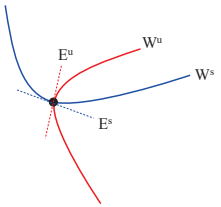
Can we use some of those cool properties and turn them to advantage?

e.g., dense attractor coverage?

or the manifold structure?

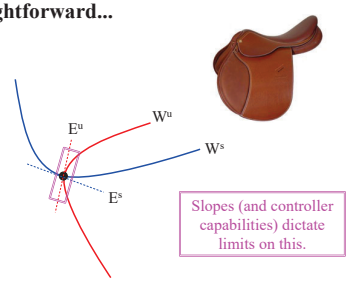
160

Remember those λ_i & un/stable manifolds? They're going to play a critical role here…



161

Balancing a system at a saddle point is fairly straightforward...

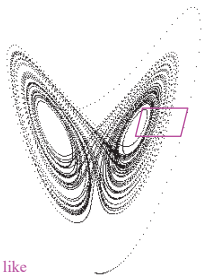


Slopes (and controller capabilities) dictate limits on this.

162

Remember these things?

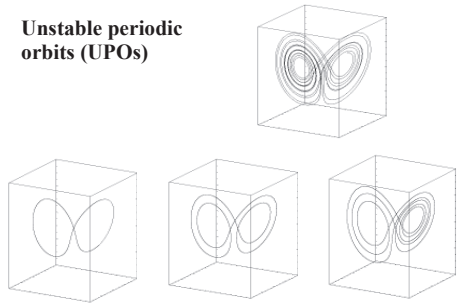
- exponential divergence of neighboring trajectories
- *often* fractal
- covered densely by trajectories
- contain an infinite number of "unstable periodic orbits"…



Sliced crosswise, these look like the picture on the previous slide!

163

Unstable periodic orbits (UPOs)



Bradley/Mantilla, *Chaos* **12**:596

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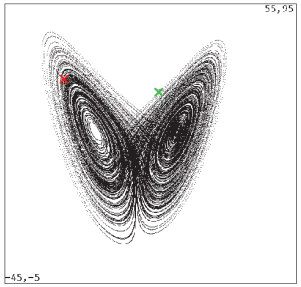
OGY control

- dense attractor coverage → reachability
- un/stable manifold structure + local-linear control → controllability

Ott *et al.*, *PRL* **64**:1196

165

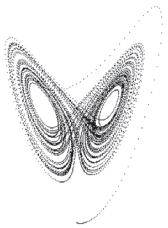
Use local-linear control, designed using the eigen-values and eigen-vectors at that point × to balance a chaotic system on a UPO passing through that point.



But you're relying on denseness to get you into the controllable region, and that may take a while…

166

Chaos and control



Can we use some of those cool properties and turn them to advantage?

e.g., dense attractor coverage?

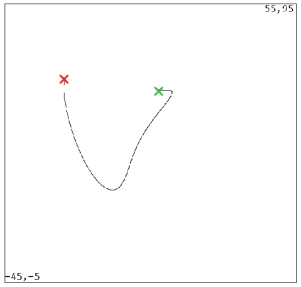
Or the manifold structure?

And what about that SDOIC business?

167

Lorenz System:

SDOIC-based targeting



Four R switches; 240X faster

Bradley, *Cybernetics & Systems* **26**:299

168

Program in Applied Mathematics

Erik Boltt

University of Colorado at Boulder
Boulder CO 80309-0526
(303) 492-4668

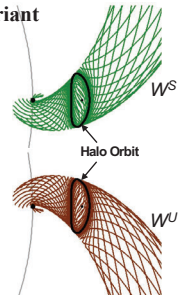
169

Other cool ways to use invariant manifolds

Want to get a spacecraft onto a "halo orbit," which is a UPO of the dynamics.

Unstable Periodic Orbits (UPOs) have invariant manifolds:

- Stable Invariant Manifold (W^S)
  - *The set of all trajectories a particle could use to arrive onto the UPO.*
- Unstable Invariant Manifold (W^U)
  - *The set of all trajectories a particle could take after a small perturbation from the UPO.*

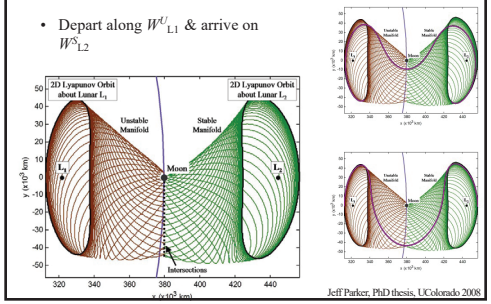


Jeff Parker, PhD thesis, UColorado 2008

170

Low-energy (cheap) orbit transfers

- Depart along W^U_{L1} & arrive on W^S_{L2}

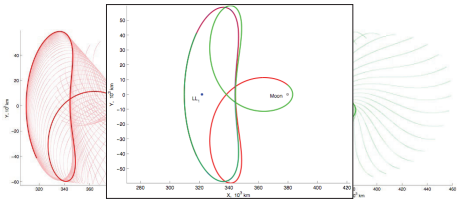


Jeff Parker, PhD thesis, UColorado 2008

171

Homoclinic orbits - The best case

- If the stable and unstable manifolds intersect—aka homoclinic connection:



Unstable Manifold of an LL_1 Lyapunov Orbit

Stable Manifold of an LL_1 Lyapunov Orbit

Jeff Parker, PhD thesis, UColorado 2008

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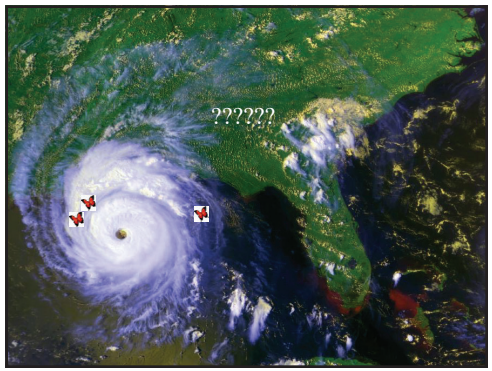
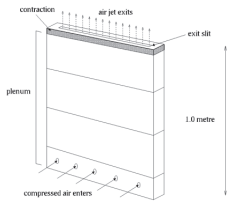
Can we do any of that in spatially extended systems?

173



174

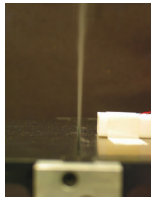
A 2D jet



Peacock *et al., Exp. Fluids* **37**:22

175

End view



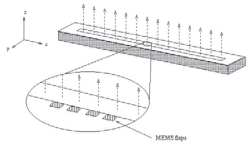
room lighting

stop-action laser "slice"

aerosolized canola oil

176

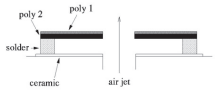
Forcing the jet flow

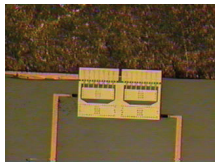


Slit: 2.5 X 400 mm

177

MEMS actuators





Video : overhead view at 2Hz, 10V

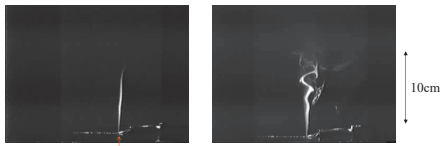
Area of individual flap is 1.0 x 0.25mm

Ma *et al., IEEE Trans. Adv. Packaging* **26**:268

178

The Butterfly effect in action…

no forcing *6Hz forcing*



MEMS flap

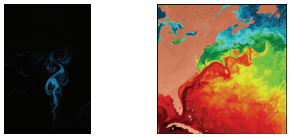
Re ~ 25

Forcing generates coherent structures that enhance entrainment and mixing

(*Exp.Fluids* 37:22)

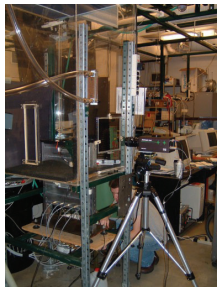
179

Does this have anything to do with reality?



180

Measurement & isolation:



181

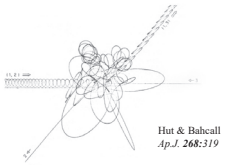
Another interesting application: chaos in the solar system

- orbits of Pluto, Mars
- Kirkwood gaps
- rotation of Hyperion & other satellites
- …

182

Solar system stability:

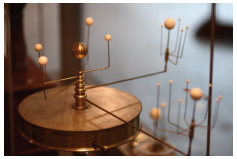
- recall: two-body problem not chaotic
- but three (or more) can be…



Hut & Bahcall
Ap.J. ***268:****319*

183

Exploring that issue before the digital computer age…



An *orrery*, which is a *mechanical computer* whose gear ratios and circular platters simulate the orbits of the planets

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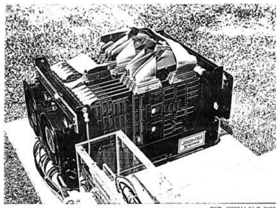
Exploring that issue, circa 1980:

- write the *n*-body equations for the solar system
- solve them using special ODE solvers on a special-purpose computer

The *digital orrery*
(Wisdom & Sussman)



ONR N00014-86-K-0180

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Numerical Evidence That the Motion of Pluto Is Chaotic

Gerald Jay Sussman and Jack Wisdom

The Digital Orrery has been used to perform an integration of the motion of the outer planets for 845 million years. This integration indicates that the long-term motion of the planet Pluto is chaotic. Nearby trajectories diverge exponentially with an *e*-folding time of only about 20 million years.

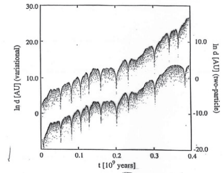


Figure 3: The exponential divergence of nearby trajectories is indicated by the average linear growth of the logarithms of the distance measures as a function of time. In the upper trace we see the growth of the variational distance around a reference trajectory. In the lower trace we see how two Plutos diverge with time. The distance saturates near 80AU when the Plutos are on opposite sides of the Sun. The variational method of studying neighboring trajectories does not have the problem of saturation. Note that the two methods are in excellent agreement until the two-trajectory method has nearly saturated.

Science **241**:433

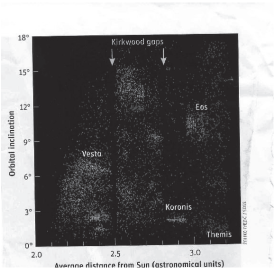
186

Should we worry?

- No.

187

Kirkwood gaps:



From *Sky & Telescope*

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Chaos and the Kirkwood gaps

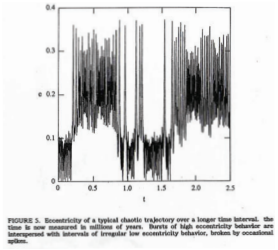

FIGURE 5. Eccentricity of a typical chaotic trajectory over a longer time interval, the time is now measured in millions of years. Bursts of high eccentricity behavior are interspersed with intervals of irregular low eccentricity behavior, broken by occasional spikes.

Wisdom, *Nuclear Phys. B* **2**:391

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Evidence in favor of the conjecture:

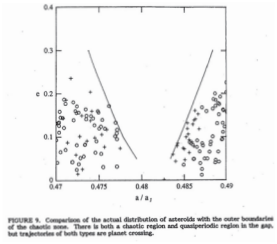


FIGURE 9. Comparison of the actual distribution of asteroids with the outer boundaries of the chaotic zone. There is both a chaotic region and quasiperiodic region in the gap, but trajectories of both types are planet crossing.

Wisdom, *Nuclear Phys. B* **2**:391

190

Chaotic tumbling of satellites:

Voyager and Galileo saw this…

From *Sky & Telescope*

Ap. J. **97**:570
Ap. J. **98**:1855

191

…so did Cassini:

www.nasa.gov/mission_pages/cassini/multimedia/pia06243.html

192

Chaotic tumbling of satellites:

This happens for **all** satellites at some point in their history, unless they are perfectly spherical and in perfectly circular orbits (pf: KAM theorem; see Wisdom *Nuc. Phys. B* **2**:391)

Some of them are still tumbling chaotically because of their geometry, but most (like the earth and its moon) have settled down into tidal equilibria or other calmer behaviors

193

More chaos in the solar system:

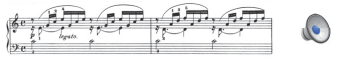
- obliquity of Mars (Touma & Wisdom, *Science* **259**:1294)

www.solarviews.com

- etc.

194

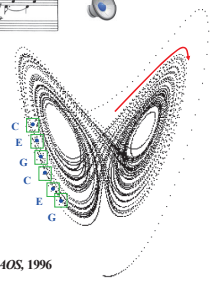
Musical Variations from a Chaotic Mapping



Pitch sequence:
C, E, G, C, E, G, C, E…

symbol dynamics

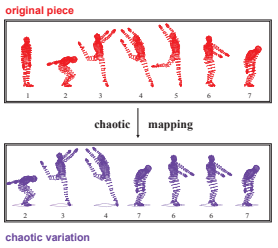
variation!



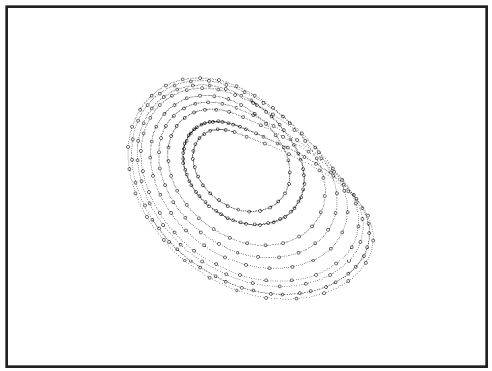
Diana Dabby, *CHAOS*, 1996

195

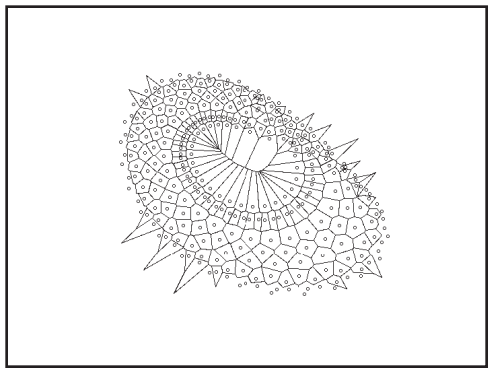
Chaotic variations on movement sequences



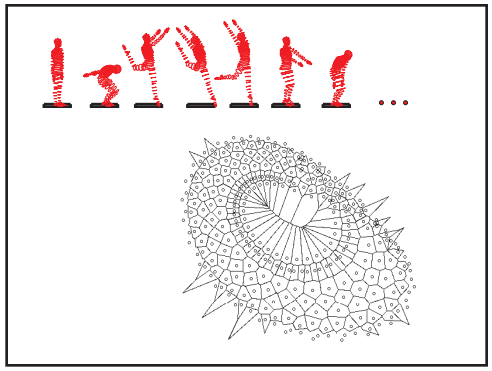
196



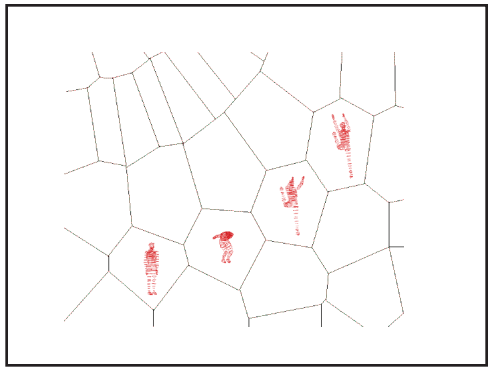
197



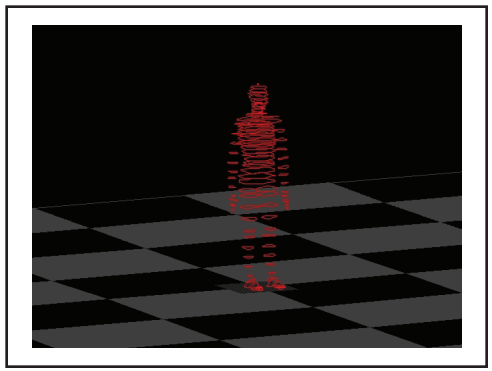
198



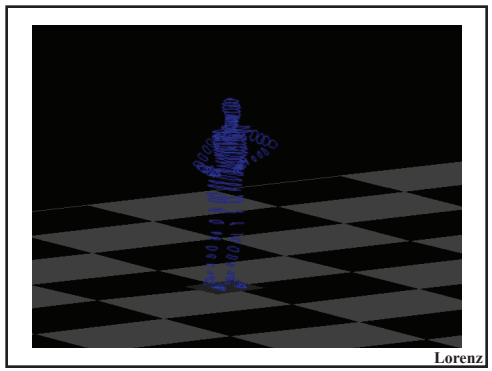
199



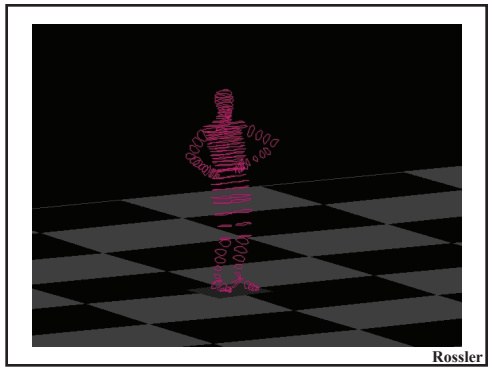
200



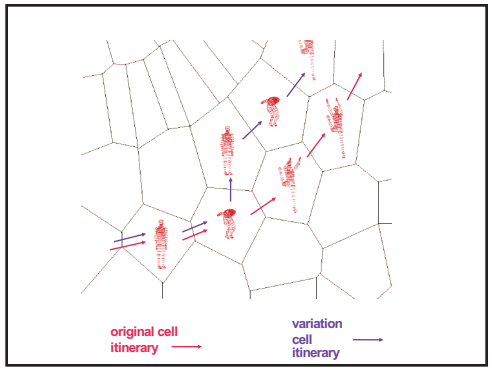
201



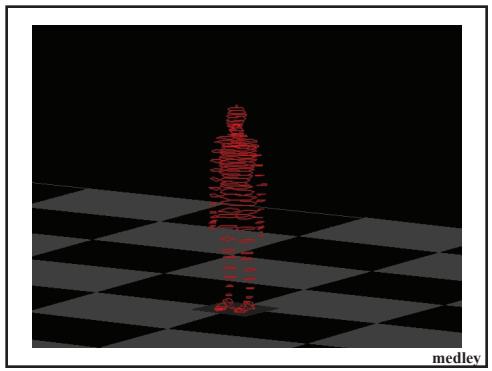
202



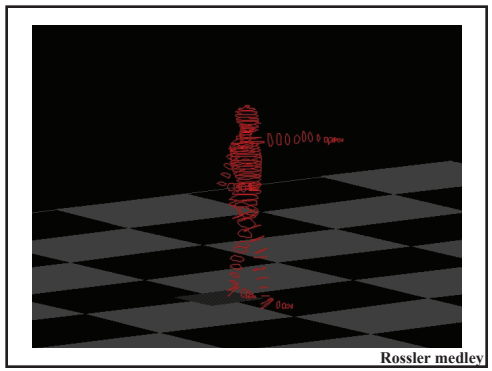
203



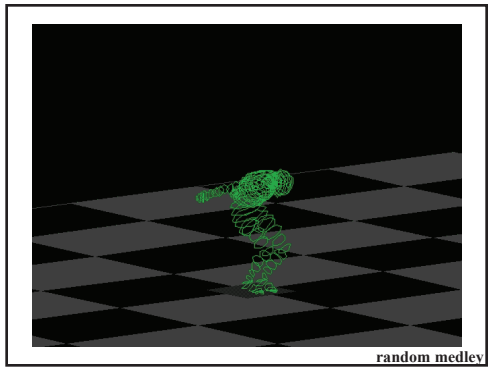
204



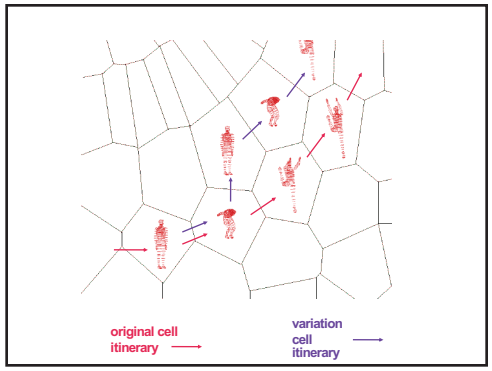
205



206



207



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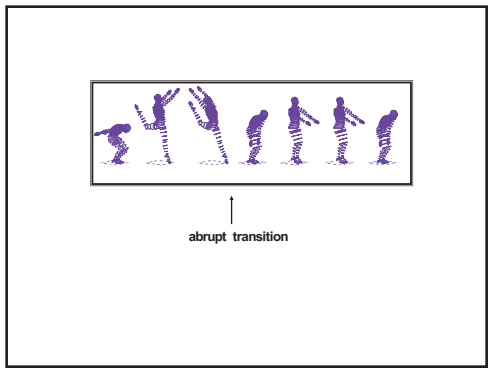

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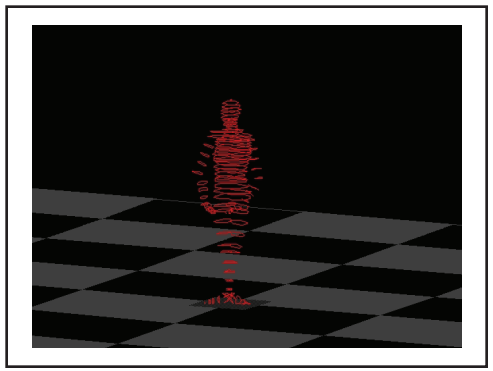

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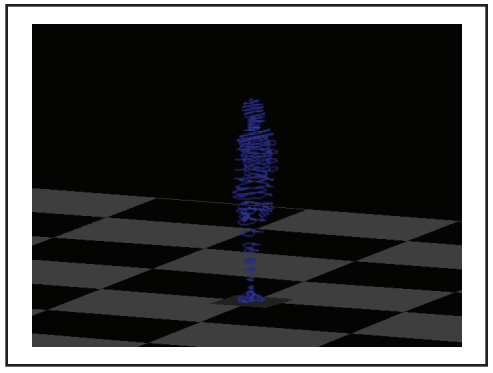


211

What this does to text

Alice was beginning to get very tired of sitting by her sister on the bank, and of having nothing to do: once or twice she had peeped into the book her sister was reading, but it had no pictures or conversations in it, `and what is the use of a book,' thought Alice `without pictures or conversation?' …..

chaotic mapping

about stopping herself she found very tired of sitting by her sister on the bank, and of having nothing to do: once or twice she had peeped into either | question, | it didn't much matter which way she put it. She | felt that she was dozing off, and had just begun to dream that she was walking hand in | hand with Dinah, and saying to her very earnestly, `Now, Dinah, tell me me the truth: did you ever ever eat a bat?'

(| symbols inserted to show the shuffle breaks)

212

This "chunkwise shuffle" mechanism is used by human composers, too:

- Lukas Foss's *Baroque Variations*
- Sampling (Big Audio Dynamite, hip hop, …)

But abrupt transitions raise different issues in movement…

ouch ouch

Original

1 2 3 4 5 6 7 8 … 98 99 100

Variation

6 7 8 | 17 18 19 20 21 | 8 9 …

213

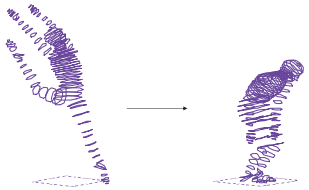
"Cutting and pasting is the essence of what hip-hop culture is all about for me. It's about drawing from what's around you, and subverting it and decontextualizing it." DJ Shadow[61].

"I look at all the different parts and see how I can organize them in a way. It's like maths. Very mathematic. It's like graphs!" Blockhead[62].

[61] http://to-the-quick.binghamton.edu/issue%202/sampling.html
[62] http://www.trip-hop.net/interview-10-Blockhead.html

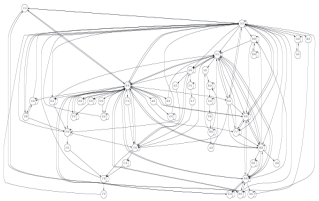
214

Interpolation



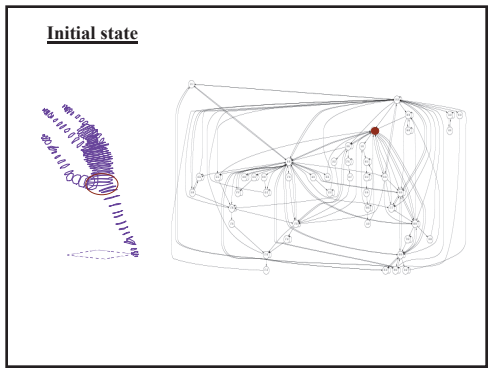
215

Corpus-based approach



- **graph captures motions of one joint**
- **note: specific to the genre of the corpus!**

216



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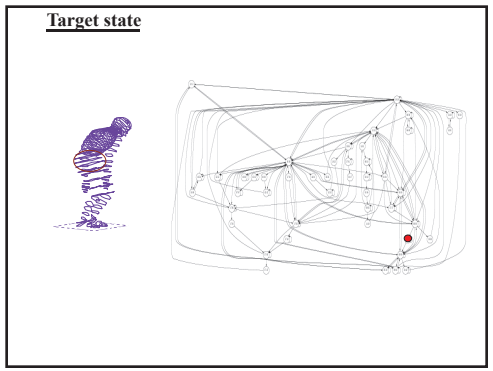

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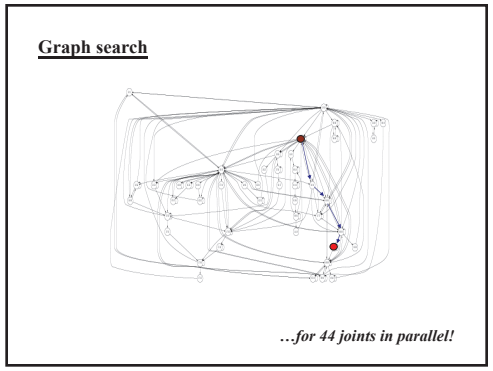

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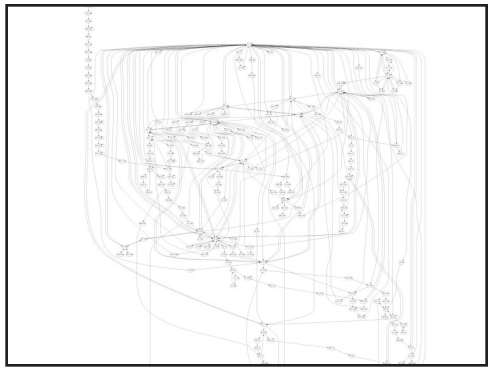

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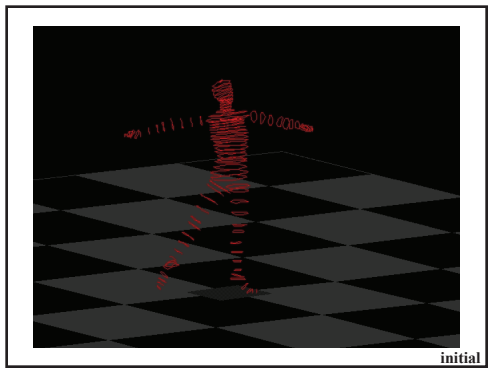

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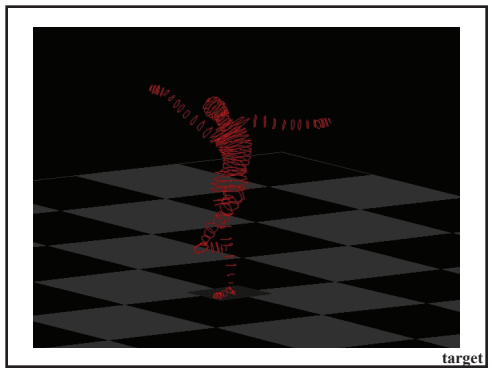

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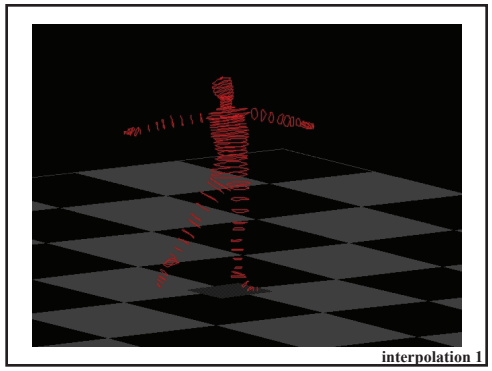

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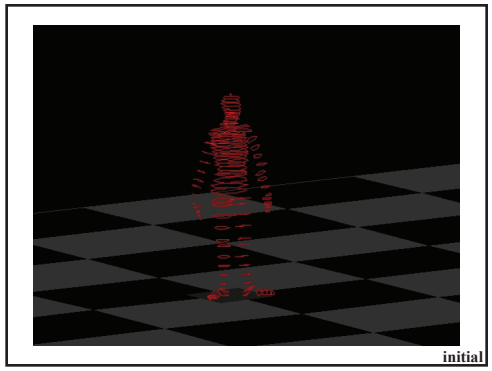

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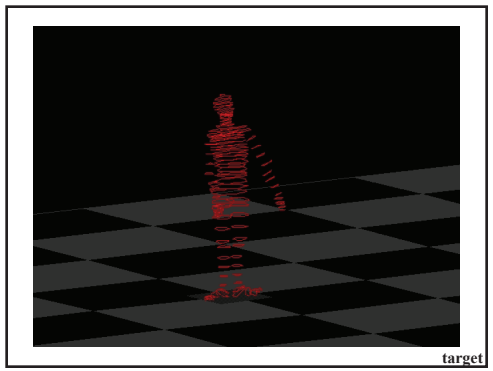

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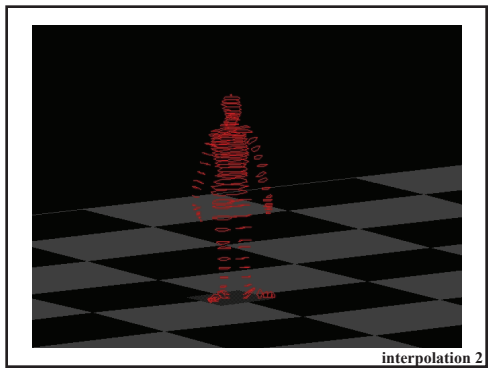

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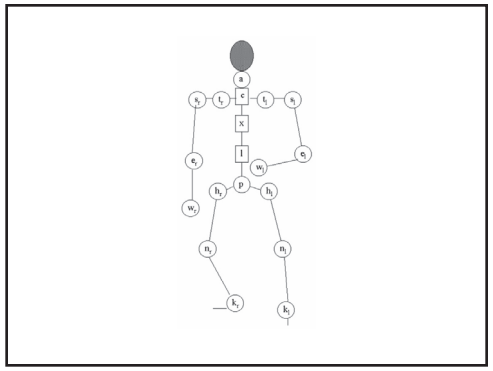

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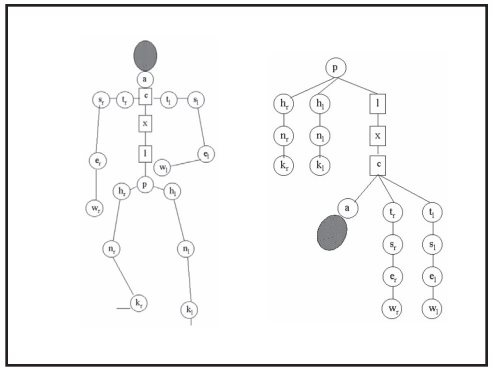

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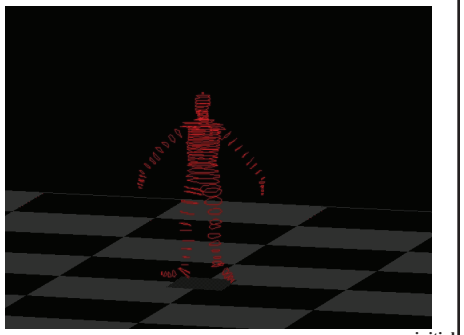

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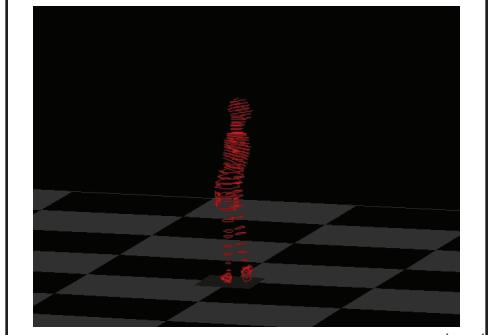

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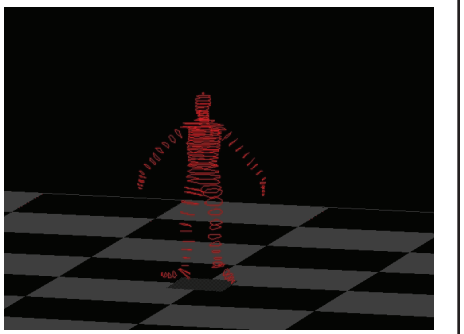


231

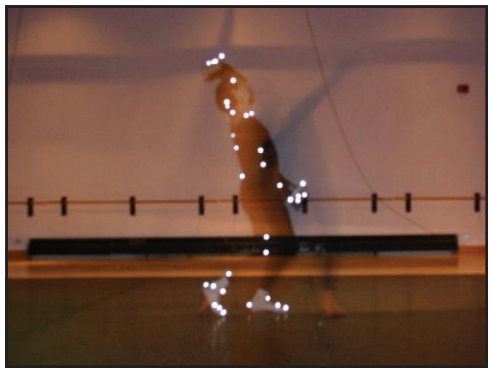
"Chaographer" and "MotionMind"

- "stylistically consonant" movement sequences
  - variations
  - interpolations
- but meager corpus can create discursive paths
- removing constraints induced by topology and gravity: ballet → modern??
- *great* way to engage laymen/students in math, physics, computer science…

Other applications:

- flight simulators
- training (e.g., wargames)
- etc.

232



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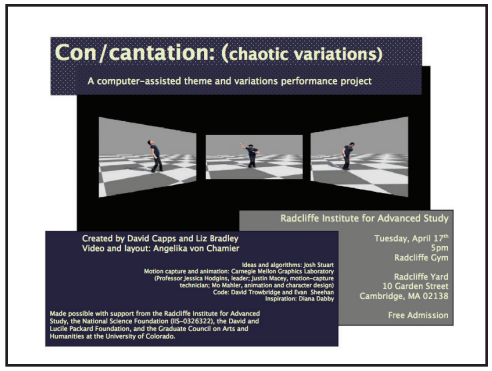

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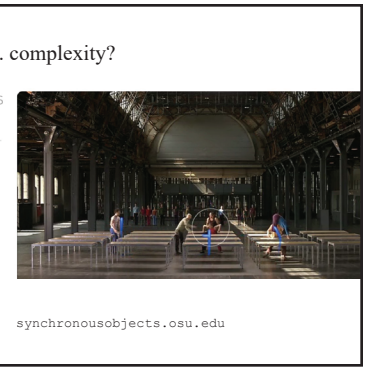

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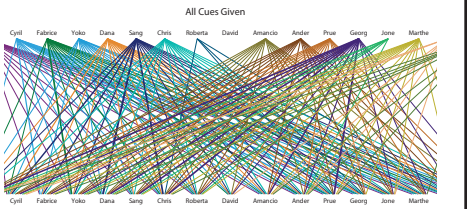

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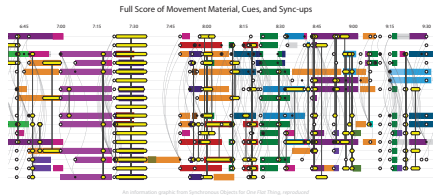

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