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Linear algebra and stability in dynamical systems

SFI - CSSS

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Diagonalising matrices

A matrix is a linear operator mapping a vector space onto another vector space (or itself).

$$\mathbf{v} \to \mathbf{w} = M\mathbf{v}$$

A square non-singular matrix can be viewed as a change of coordinates (rotation and scaling) in a vector space. Such a change of coordinates (defined by a matrix M**) affects vectors (v) and matrices (**A**) as:**

$$\mathbf{v} \to \tilde{\mathbf{v}} = M\mathbf{v}$$

$$A \to \tilde{A} = MAM^{-1}$$ **(similarity transformation)**

Which can be understood e.g. by considering the linear equation:

$$\mathbf{u} = A\mathbf{v}$$

$$M\mathbf{u} = MA\mathbf{v} = MAM^{-1}M\mathbf{v}$$

$$\tilde{\mathbf{u}} = \tilde{A}\tilde{\mathbf{v}}$$

Diagonalising matrices, cont.

Typically, there exists a similarity transformation T **such that:**

$$A \to TAT^{-1} = D$$

where D **is a diagonal matrix.** **What is at the structure of** T**?**

Recall eigenvalues and eigenvectors:

$$A\mathbf{v}_i = \lambda_i \mathbf{v}_i$$ **right eigenvectors**

$$\mathbf{u}_i^T A = \lambda_i \mathbf{u}_i^T$$ **left eigenvectors**

$$\mathbf{v}_i = \mathbf{u}_i$$ **for symmetric matrices**

$$\boxed{\mathbf{u}_i \cdot \mathbf{v}_j = 0 \text{ if } \lambda_i \neq \lambda_j} \text{(calculate } \mathbf{u}_i^T A \mathbf{v}_j \text{ in two different ways...)}$$

Diagonalising matrices, cont.

$$A \to TAT^{-1} = D$$

$$\underbrace{\begin{pmatrix} - & \mathbf{u}_1^T & - \\ - & \mathbf{u}_2^T & - \\ & \vdots & \end{pmatrix}}_{T} A \underbrace{\begin{pmatrix} | & | & \\ \mathbf{v}_1 & \mathbf{v}_2 & \cdots \\ | & | & \end{pmatrix}}_{T^{-1}} = \begin{pmatrix} - & \mathbf{u}_1^T & - \\ - & \mathbf{u}_2^T & - \\ & \vdots & \end{pmatrix} \begin{pmatrix} | & | & \\ \lambda_1\mathbf{v}_1 & \lambda_2\mathbf{v}_2 & \cdots \\ | & | & \end{pmatrix} \overset{\substack{\mathbf{u}_i \cdot \mathbf{v}_j = 0 \text{ if } \lambda_i \neq \lambda_j \\ \big\downarrow}}{=} \underbrace{\begin{pmatrix} \lambda_1 & 0 & \cdots \\ 0 & \lambda_2 & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}}_{D}$$

Comment on exceptions

There are matrices that cannot be diagonalised. For example:

$$\begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix}$$

which has eigenvalue a **but only one eigenvector and cannot be diagonalised. It is an example of a Jordan form, which can be generalised. However, these matrices are singular exceptions because**

$$\begin{pmatrix} a & 1 \\ 0 & a+\epsilon \end{pmatrix} \text{ where } \epsilon \neq 0$$

can be diagonalised, but the eigenvectors are close to linearly dependent as $\epsilon \to 0$

Linear dynamical systems

Consider $\dot{\mathbf{x}} = A\mathbf{x}$

Diagonalise A **by the change of variables** y=Tx

$$\dot{y} = \dot{T\mathbf{x}} = T\dot{\mathbf{x}} = TA\mathbf{x} = TAT^{-1}T\mathbf{x} = D\mathbf{y}$$ ➡ $$\dot{y}_i = \lambda_i y_i \Rightarrow$$ $$y_i = c_i \exp(\lambda_i t)$$

i.e.

$$\mathbf{y} = \begin{pmatrix} c_1 \exp(\lambda_1 t) \\ c_2 \exp(\lambda_2 t) \\ \vdots \end{pmatrix}$$

c_i determined by initial conditions:$\sum_i c_i \mathbf{v}_i = \mathbf{x}(0)$

$$\boxed{\mathbf{x} = T^{-1}\mathbf{y} = \begin{pmatrix} | & | & \\ \mathbf{v}_1 & \mathbf{v}_2 & \dots \\ | & | & \end{pmatrix} \begin{pmatrix} c_1 \exp(\lambda_1 t) \\ c_2 \exp(\lambda_2 t) \\ \vdots \end{pmatrix} = \sum_i c_i \mathbf{v}_i \exp(\lambda_i t)}$$

Stability

So, it follows that the dynamical system

$$\dot{\mathbf{x}} = A\mathbf{x}$$

has the solution

$$\mathbf{x} = \sum_i c_i \mathbf{v}_i \exp(\lambda_i t)$$

time scales

and therefore

$$\mathbf{x} \to 0 \text{ as } t \to \infty \text{ if the real part of } \lambda_i \text{ are negative for all eigenvalues}$$

Intuitively we may say that the system is stable since, if it is at its fixed point and there is a perturbation, the deviation will decay exponentially and the system will return to its resting state.

BUT WE NEED TO BE CAREFUL HERE!!!

A complication

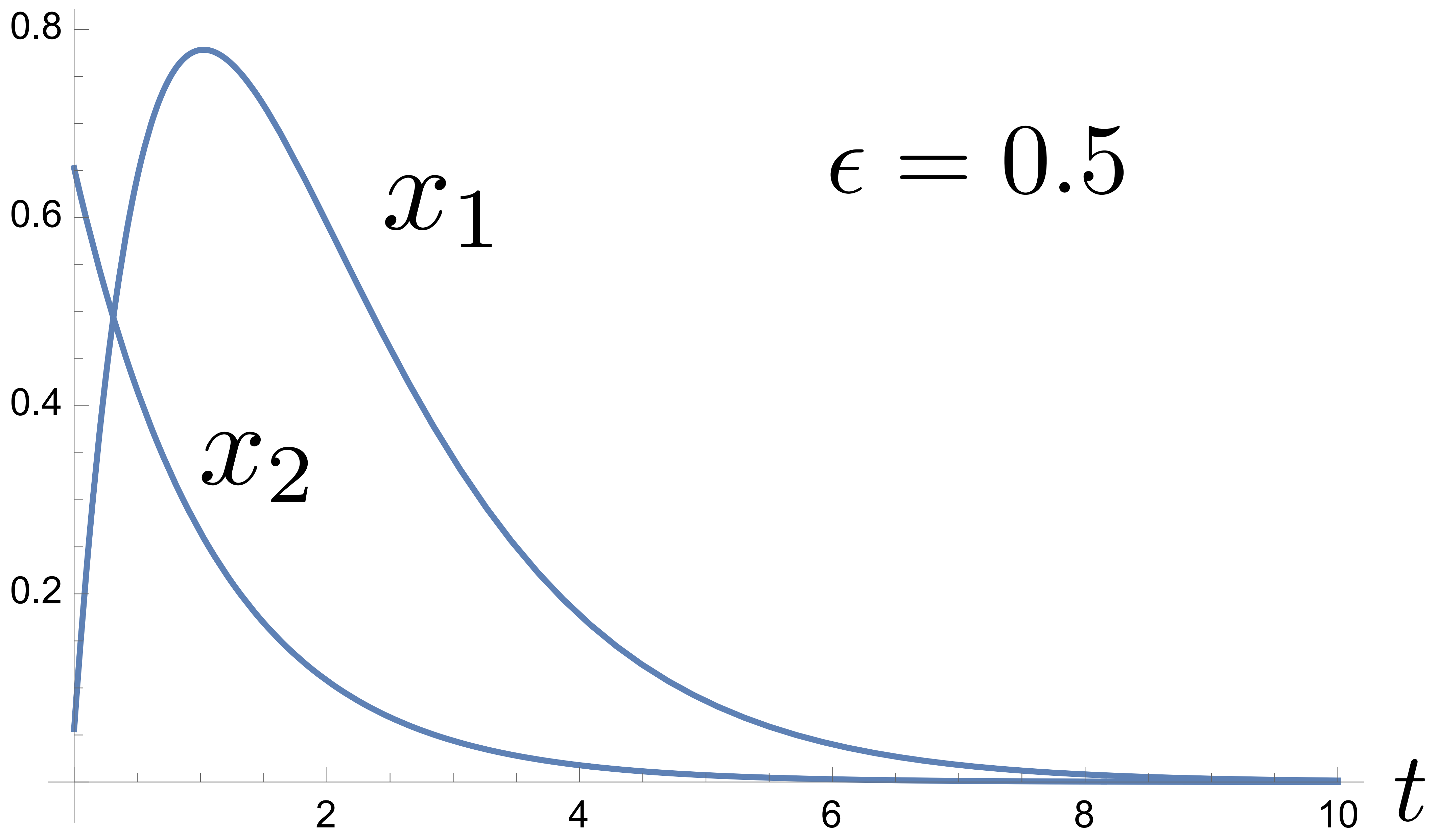
Consider

$$A = \begin{pmatrix} -1 & 3 \\ 0 & -1+\epsilon \end{pmatrix}$$

with eigenvalues

$$\lambda_1 = -1 \qquad \lambda_2 = -1+\epsilon$$

clearly, if ϵ is small, both eigenvalues have negative real part.



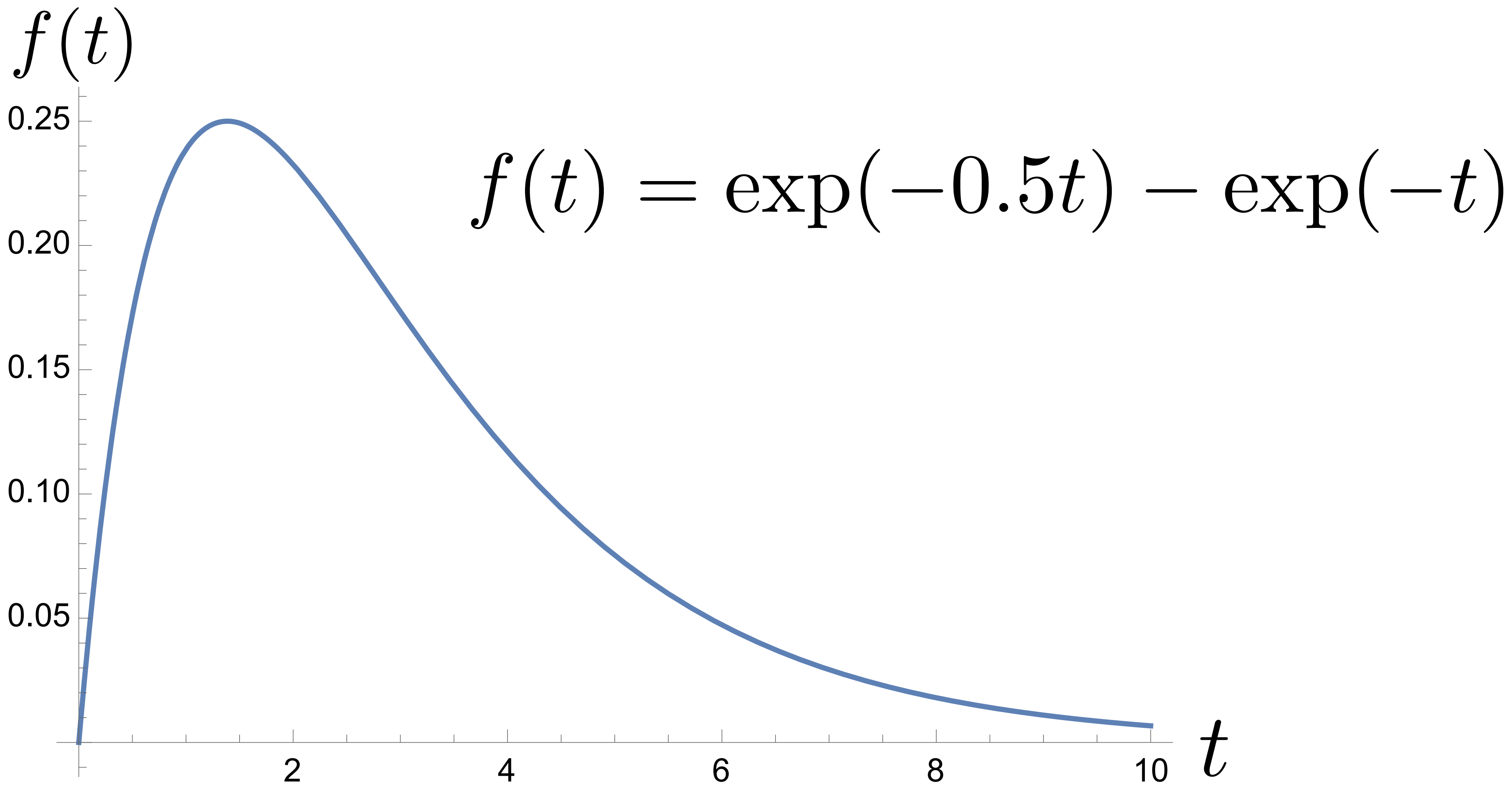
So, while it is true that the system converges to zero given enough time, there can be large deviations before this happens…

Why does this happen?

The answer is actually very simple. The solution is a sum of exponentials

$$\mathbf{x} = \sum_i c_i \mathbf{v}_i \exp(\lambda_i t)$$

But even if every eigenvalue is negative, so the amplitude of each term shrinks, the sum can still grow because the terms can have "different signs" and cancel at t=0. As an illustration, look at



$$f(t) = \exp(-0.5t) - \exp(-t)$$

Why it matters

Linear systems are often used to understand the (local) behaviour of non-linear systems. Consider

$$\dot{\mathbf{x}} = f(\mathbf{x}) \text{ where } f : \mathbb{R}^n \to \mathbb{R}^n \text{ and } f(\mathbf{x}_0) = 0$$

If the system is close to its fixed point and f is “smooth enough”, then we can linearise the equations and study the local behaviour through

Jacobian

$$\dot{\delta\mathbf{x}} = J_{\mathbf{x}_0}\delta\mathbf{x} + \mathcal{O}(\delta x^2) \text{ where } J_{\mathbf{x}_0} = \left.\frac{\partial f_i(\mathbf{x})}{\partial x_j}\right|_{\mathbf{x}=\mathbf{x}_0}$$

$$\delta\mathbf{x} = \mathbf{x} - \mathbf{x}_0$$

So, the stability of a fixed point in a non-linear system can be studied through the linearised equations defined by the Jacobian of the system at the fixed point. But then, large deviations can be very problematic since the linearisation itself may break down…

Stability in linear systems revisited

Consider (again) $\dot{\mathbf{x}} = A\mathbf{x}$

We would like a condition that ensures deviations to decay monotonically with time, i.e.

deviation

$$\frac{d(\mathbf{x}\cdot\mathbf{x})}{dt} < 0 \quad \Longrightarrow \quad \boxed{\mathbf{x}^T A\mathbf{x} < 0}$$

Since $\mathbf{x}^T A\mathbf{x}$ is a number, $\mathbf{x}^T A\mathbf{x} = (\mathbf{x}^T A\mathbf{x})^T = \mathbf{x}^T A^T\mathbf{x}$ $\Longrightarrow$ $\mathbf{x}^T A\mathbf{x} = \frac{1}{2}\mathbf{x}^T(A + A^T)\mathbf{x} \overset{\text{defining } A^s}{=} \mathbf{x}^T A^s\mathbf{x}$

Let $\mathbf{v}_i$ be eigenvectors of A^s, which are orthogonal since A^s is symmetric. $A^s\mathbf{v}_i = \mu_i\mathbf{v}_i$

Symmetry → μ_i and a_i are real

Let $\mathbf{x} = \sum_i a_i\mathbf{v}_i$, then

$$\mathbf{x}^T A^s\mathbf{x} = \left(\sum_i a_i\mathbf{v}_i^T\right) A^s \left(\sum_j a_j\mathbf{v}_j\right) = \left(\sum_i a_i\mathbf{v}_i^T\right)\left(\sum_j \mu_j a_j\mathbf{v}_j\right) = \sum_i \mu_i a_i^2 < 0 \text{ if } \mu_i < 0$$

Stability in linear systems conclusion

$$\dot{\mathbf{x}} = A\mathbf{x}$$

If all eigenvalues of A has negative real part, the system will eventually converge to the fixed point at zero, but deviations from zero may increase before it starts converging.

If all eigenvalues of $A^s = (A + A^T)/2$ are negative (they are real since A^s is symmetric, any deviation measured as $\sqrt{\mathbf{x} \cdot \mathbf{x}}$ will monotonically decrease with time. In this case A is said to be negative definite.

Note that if the matrix is symmetric in the first place, the first situation includes the second.

Re-visit our example

Consider

$$A = \begin{pmatrix} -1 & 3 \\ 0 & -1+\epsilon \end{pmatrix} \qquad \epsilon < 1$$

with eigenvalues

$$\lambda_1 = -1 \qquad \lambda_2 = -1+\epsilon$$ ← **both negative**

but

$Q = \mathbf{x}^T A\mathbf{x}$ can be positive, e.g. $x = (1,1) \Rightarrow Q = 1+\epsilon > 0$

For $\epsilon = 0.5$, A^s has eigenvalues (approx.) -4.54 and 1.54, showing that the system can have non-decreasing deviations.

Fixed point stability and Lyaponov functions

In general it is often hard to prove that a dynamical system has a fixed point that will attract trajectories in some region. One method often used to approach this problem is to try to construct a so called Lyaponov function. Here is the idea. Consider the dynamical system

$\dot{\mathbf{x}} = f(\mathbf{x})$ where $f : \mathbb{R}^n \to \mathbb{R}^n$ and $f(\mathbf{0}) = 0$

Construct a smooth scalar function $V(\mathbf{x}) : \mathbb{R}^n \to \mathbb{R}$, that decrease with time:

$$\dot{V} \quad = \quad \nabla V(\mathbf{x}) \cdot f(\mathbf{x}) < 0$$

when $\mathbf{x} \neq \mathbf{0}$. The function V is called a Lyaponov function.

If there exist a Lyaponov function, then 0 is a (locally) stable fixed point.

Example: $V(\mathbf{x}) = \mathbf{x} \cdot \mathbf{x}$ is a Lyaponov function for a linear system iff A is negative definite, because then $\frac{d}{dt}V(\mathbf{x}) < 0$.

May’s argument

A complex systems with interacting agents (e.g. an ecosystem) can be described by a dynamical system. Consider a stationary state (a fixed point of that system), then the Jacobian looks like

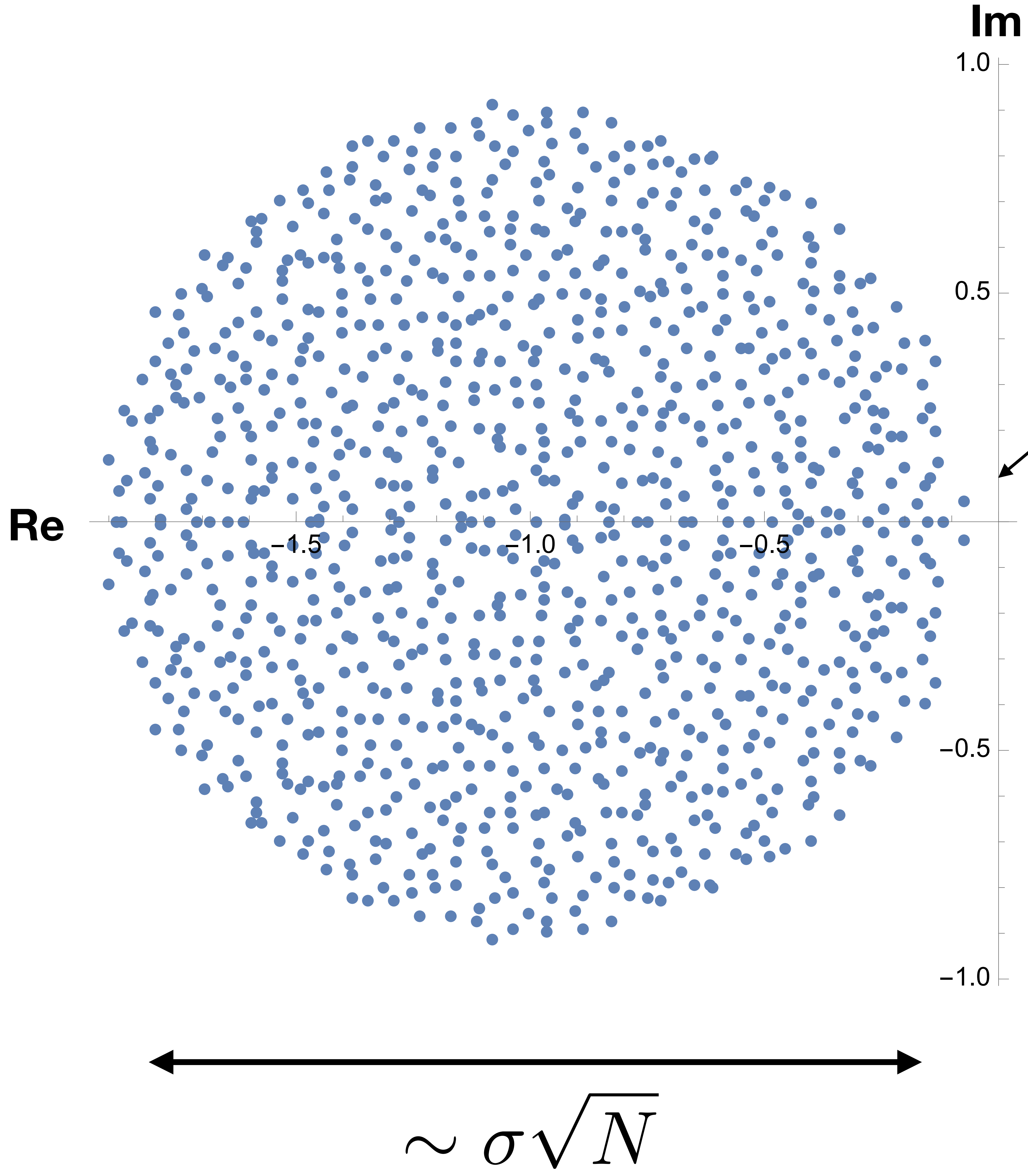
$$J = \sigma A - I$$

where A_{ij} describes the interaction between species i and j and σ is a parameter that can tune the interaction strength. The identity matrix I ensures that the system is stable in the absence of interactions.

May’s main idea was to look at the eigenvalues (spectrum) of the Jacobian and determine if the system was stable or not (all eigenvalues have negative real parts). As a model he assumed random interactions and could then use random matrix theory to estimate the spectral radius of A **and thereby also the Jacobian.**

Example of a spectrum

$$J = \sigma A - I$$



$\sim \sigma\sqrt{N}$

Critical line, $\Re(\lambda_i) < 0$

Main conclusion: when the interaction strength or the number of species increase, the system reaches a critical point and then becomes unstable.

Generalized Lotka-Volterra

$$\dot{x}_i = x_i(1 - x_i) + \sigma \sum_j A_{ij} x_i x_j$$

Self-regulation **Interaction**

Fixed points:

$$x_i(\underline{1 - x_i + \sigma \sum_j A_{ij} x_j}) = 0 \Rightarrow \begin{cases} x_i^* = 0 \\ x_i^* = 1 + \sigma \sum_j A_{ij} x_j^* \end{cases} \quad \text{or}$$

linear equation →

$$\mathbf{x}^* = (I - \sigma A)^{-1} \mathbf{1} \qquad \text{if } x_i^* > 0$$

$$\mathbf{1} = (1, 1, 1, 1 \ldots)^T$$

Jacobian

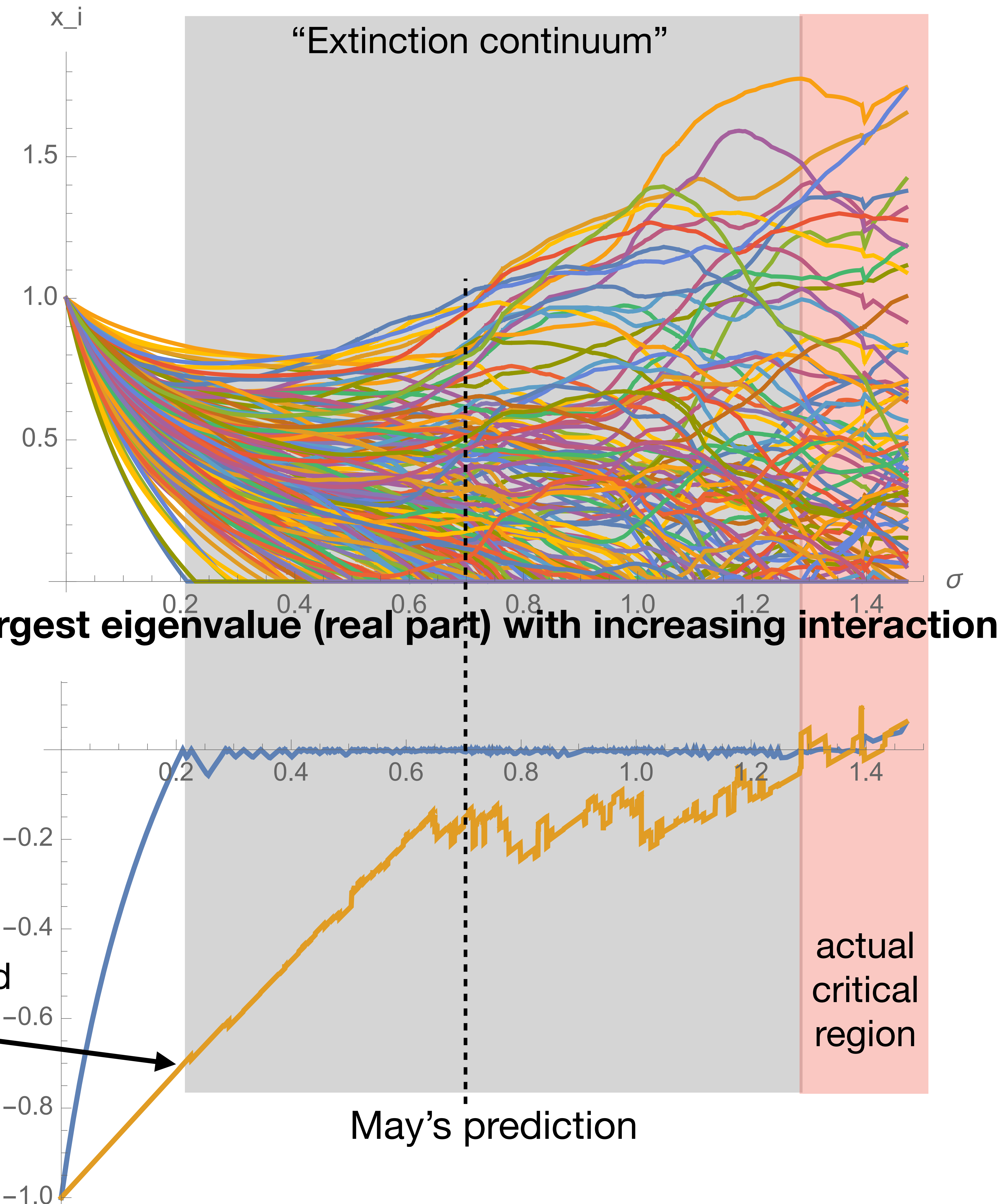
$$J = X(\underline{\sigma A - I}) \text{ where } X = \mathrm{diag}(x_1^*, x_2^*, \dots)$$

May’s Jacobian

assuming $x_i^* > 0$

depends on the fixed point, natural in interacting systems that are not typically linear

Fixed point with increasing interaction strength



Largest eigenvalue (real part) with increasing interaction

“Extinction continuum”

Eigenvalue of May’s Jacobian for reduced system

May’s prediction

actual critical region