# Introduction to Nonlinear Dynamics

Santa Fe Institute

Complex Systems Summer School

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Elizabeth Bradley

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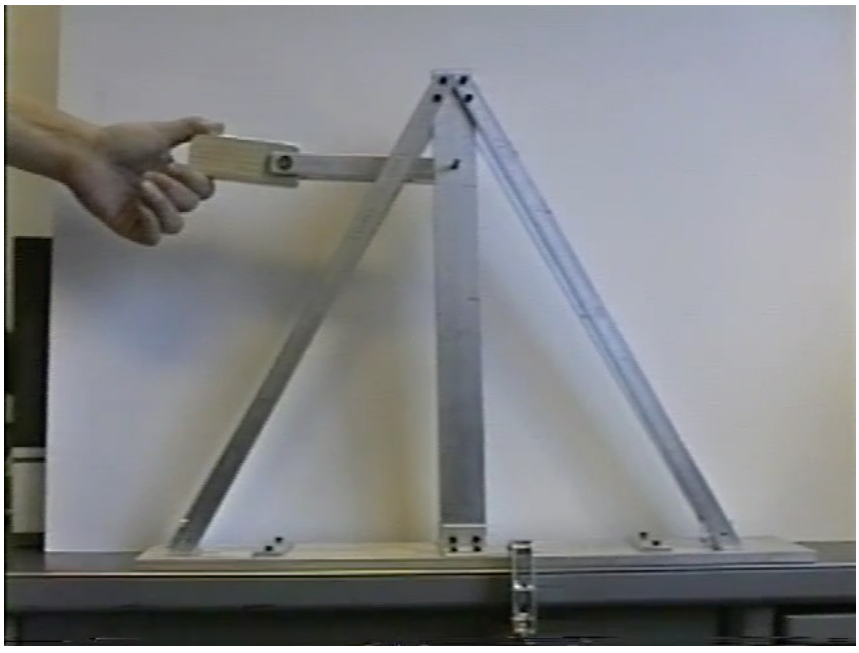

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http://ayresriverblog.com

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## Chaos

Complex behavior, arising in a (possibly quite simple!) deterministic nonlinear dynamic system, which exhibits two special properties:

- sensitive dependence on initial conditions
- characteristic structure…

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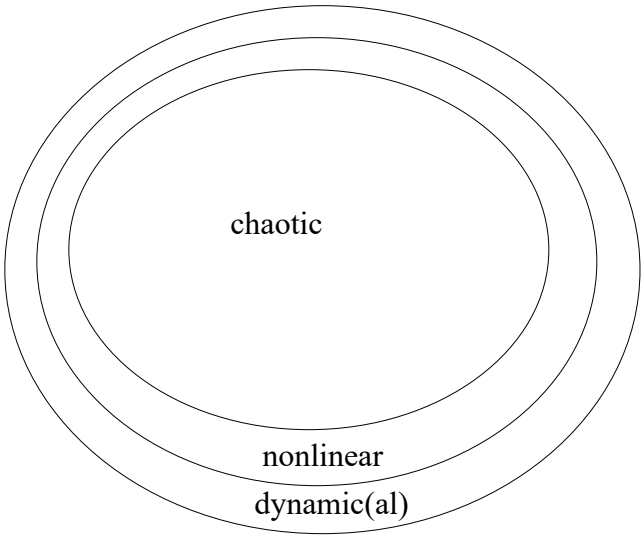
## Nonlinearity (and chaos) are ubiquitous!

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## Where nonlinear dynamics turns up

- Flows (of fluids, heat, …)
  - Eddy in creek
  - Weather
  - Vortices around marine invertebrates
  - Air/fuel flow in combustion chambers

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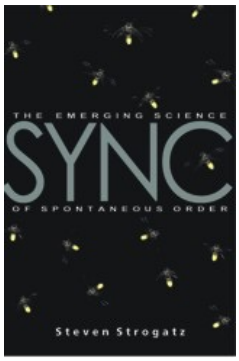
## Where nonlinear dynamics turns up

- Driven nonlinear oscillators
  - Pendula
  - Hearts
  - Fireflies




- and lots of other electronic, chemical, & biological systems

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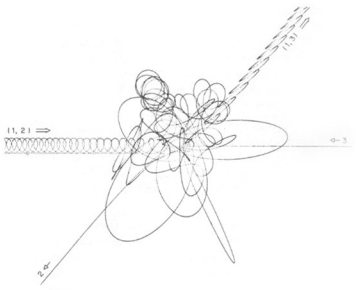
## Where nonlinear dynamics turns up

- Classical mechanics
  - three-body problem
  - paired black holes
  - pulsar emission
  - ….



Hut & Bahcall *Ap.J.* ***268:319***

- Protein folding
- Population biology
- And many, many other fields (including yours)

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- discrete time systems:
  - time proceeds in clicks
  - "maps"
  - modeling tool: difference equations
- continuous time systems:
  - time proceeds smoothly
  - "flows"
  - modeling tool: differential equations

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If $x^*$ is a fixed point of a map $f$, then $f(x^*) = x^*$

- *Stable* or *attracting* fixed point: perturbations shrink
- *Unstable* or *repelling* fixed point: perturbations grow…

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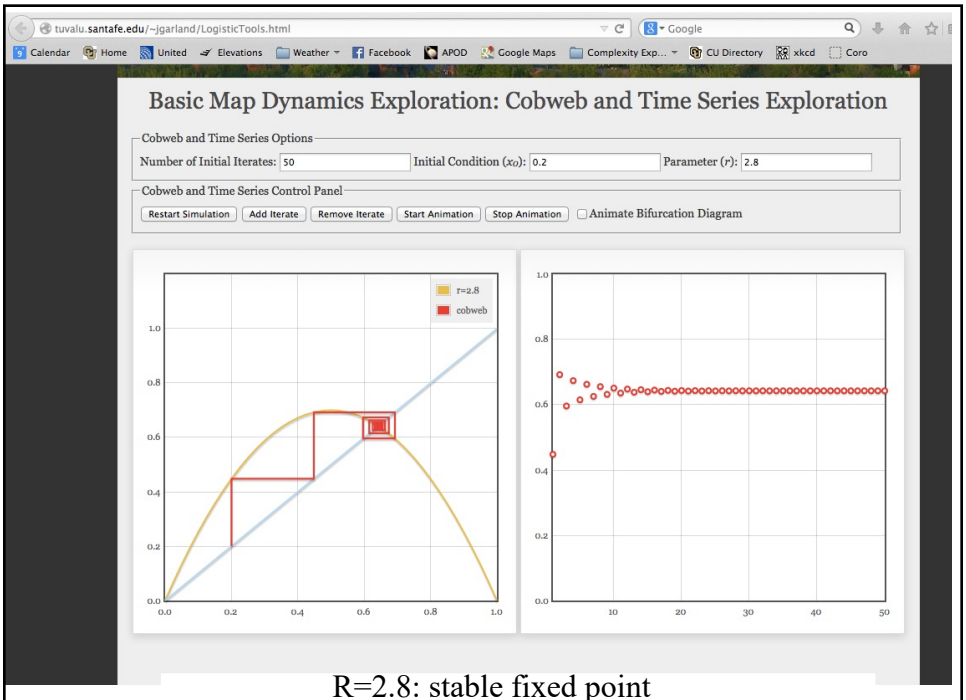


R=2.8: stable fixed point

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## This is a useful *graphical* solution/comprehension technique!

- "cobweb" diagram
- *aka* return map
- *aka* correlation plot

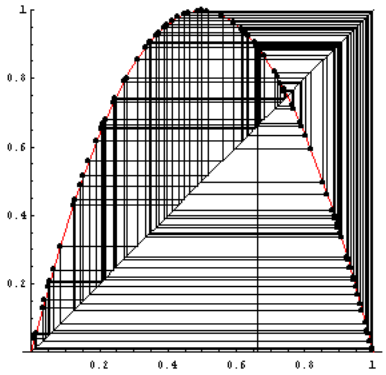


Image from Doug Ravenel's website at URochester

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## Bifurcations

Qualitative changes in the dynamics (the attractor, technically) caused by changes in *parameters:*

- Heart: pathology
- Eddy in creek: water level
- Olfactory bulb: smell
- Brain: blood chemicals
- Logistic map: R parameter…

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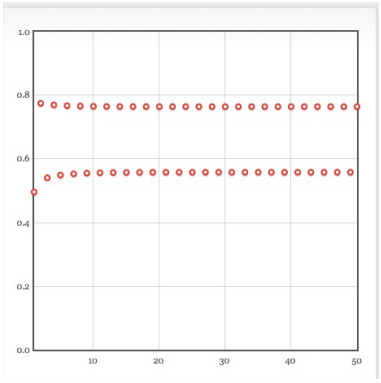
## R=3.1: two cycle!




$x_1$*
$x_2$*

Two pieces of a single attractor

(not two fixed points)

$f(f(x_1^*)) = x_1^*$
$f(f(x_2^*)) = x_2^*$

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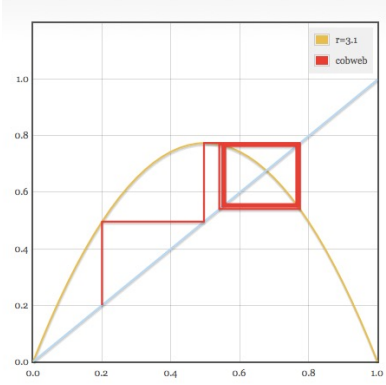
## What does that look like on a return map?




This is the system attractor for a wide range of R

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## A useful metaphor:

(image from wikipedia)

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## More broadly…


(image from wikipedia)

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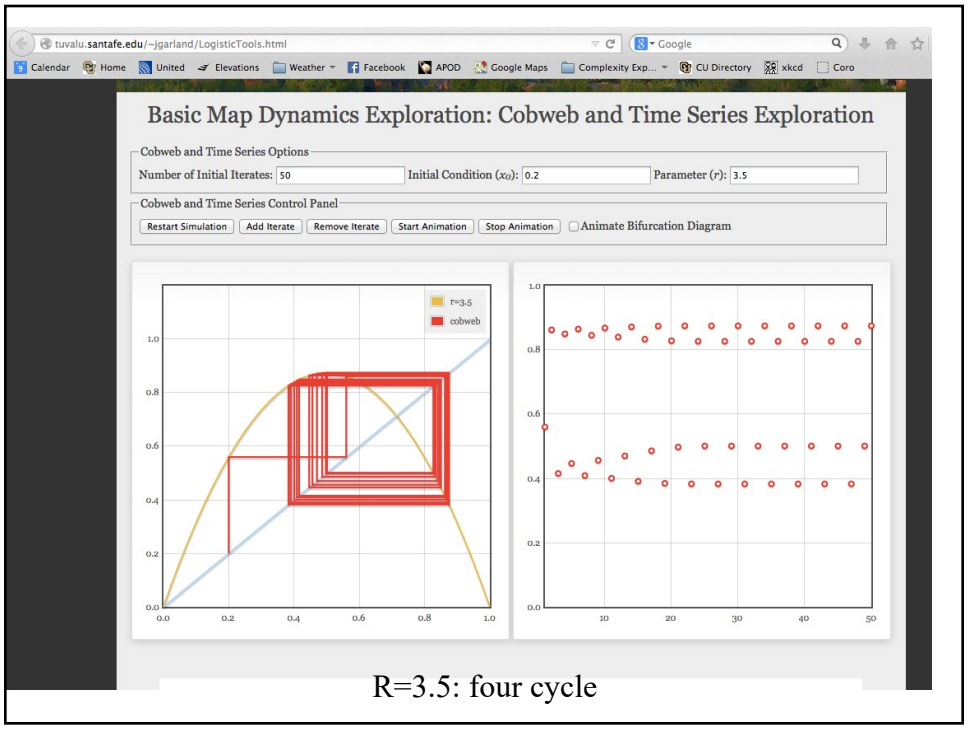


R=3.5: four cycle

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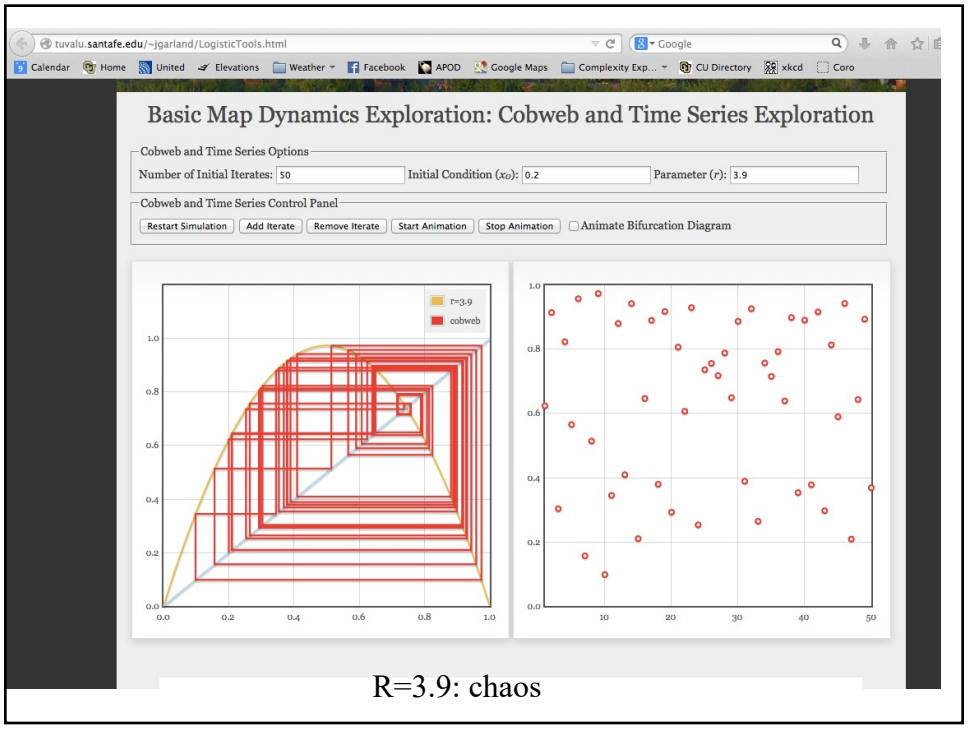


R=3.9: chaos

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## Chaos

Complex behavior, arising in a (possibly quite simple) deterministic nonlinear dynamic system, which exhibits two special properties:

- sensitive dependence on initial conditions — **SDOIC**
- characteristic structure…

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## Showing all of that on one plot: the "bifurcation diagram"…


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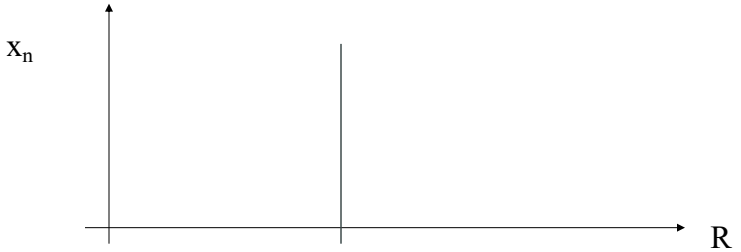


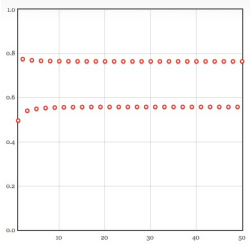

R = 3.1

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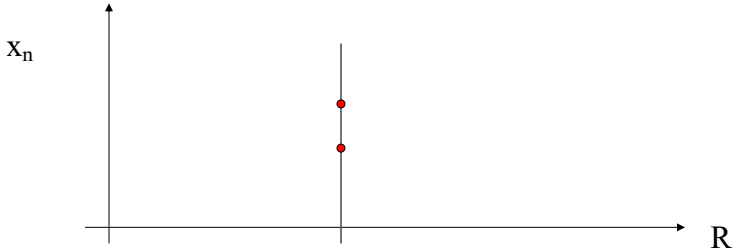
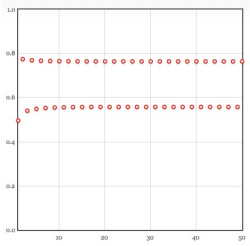
## Showing all of that on one plot: the "bifurcation diagram"…



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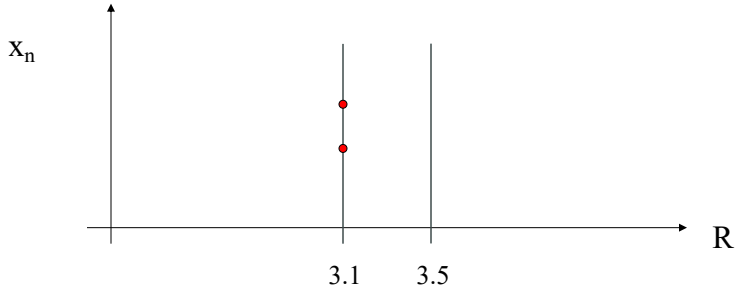


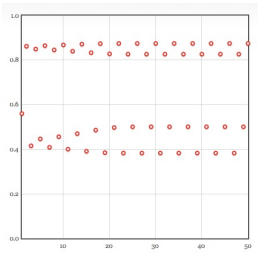

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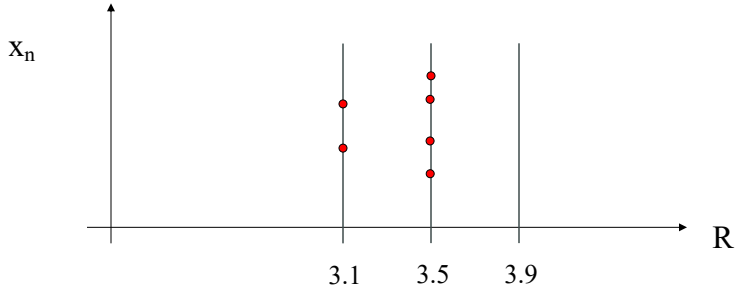


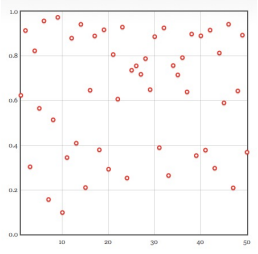

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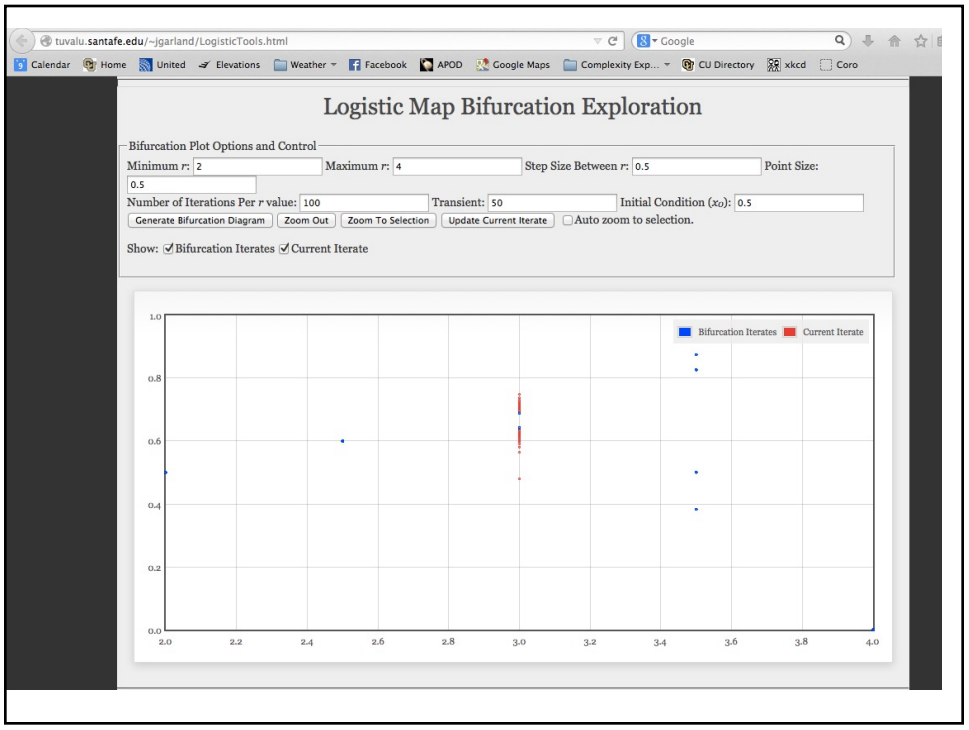
tuvalu.santafe.edu/~jgarland/LogisticTools.html
Google
Calendar
Home
United
Elevations
Weather
Facebook
APOD
Google Maps
Complexity Exp...
CU Directory
xkcd
Coro
Logistic Map Bifurcation Exploration
Bifurcation Plot Options and Control
Minimum r: 2
Maximum r: 4
Step Size Between r: 0.5
Point Size:
0.5
Number of Iterations Per r value: 100
Transient: 50
Initial Condition (x0): 0.5
Generate Bifurcation Diagram
Zoom Out
Zoom To Selection
Update Current Iterate
Auto zoom to selection.
Show: Bifurcation Iterates Current Iterate
Bifurcation Iterates
Current Iterate
1.0
0.8
0.6
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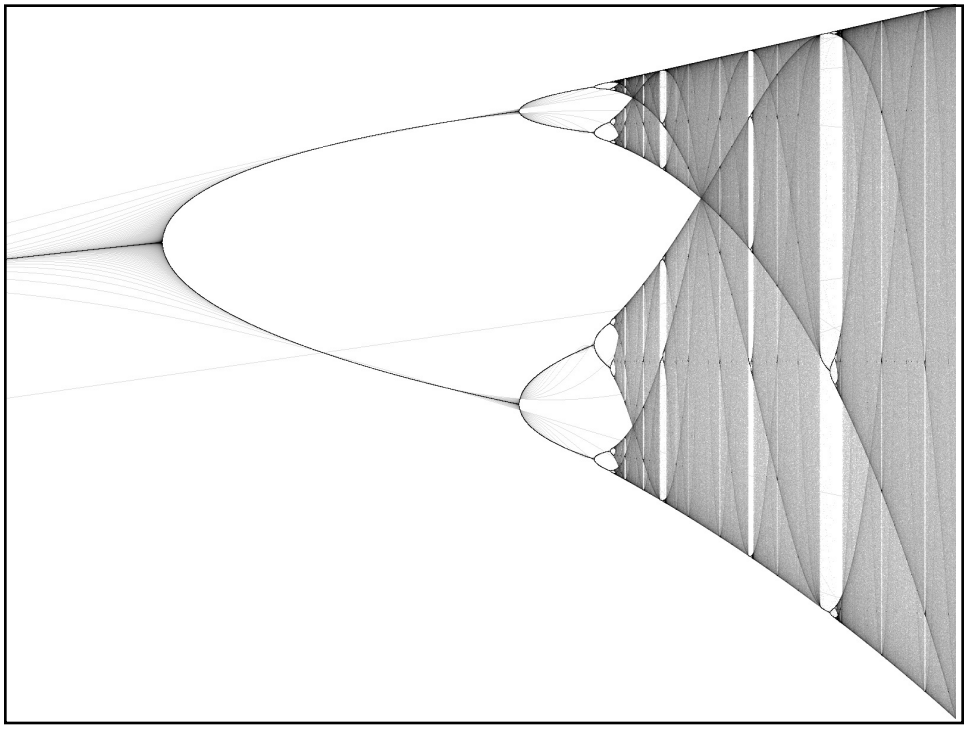

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- chaos
- veils/bands: places where chaotic attractor is dense

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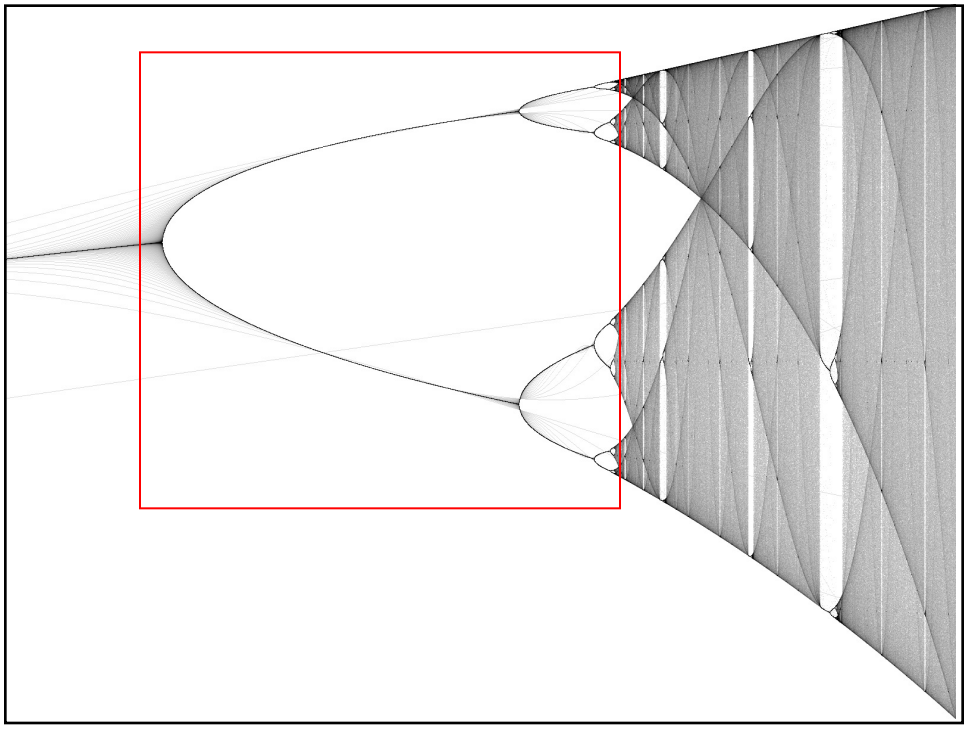

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- chaos
- veils/bands: places where chaotic attractor is dense (UPOs)
- *period-doubling cascade @ low R*

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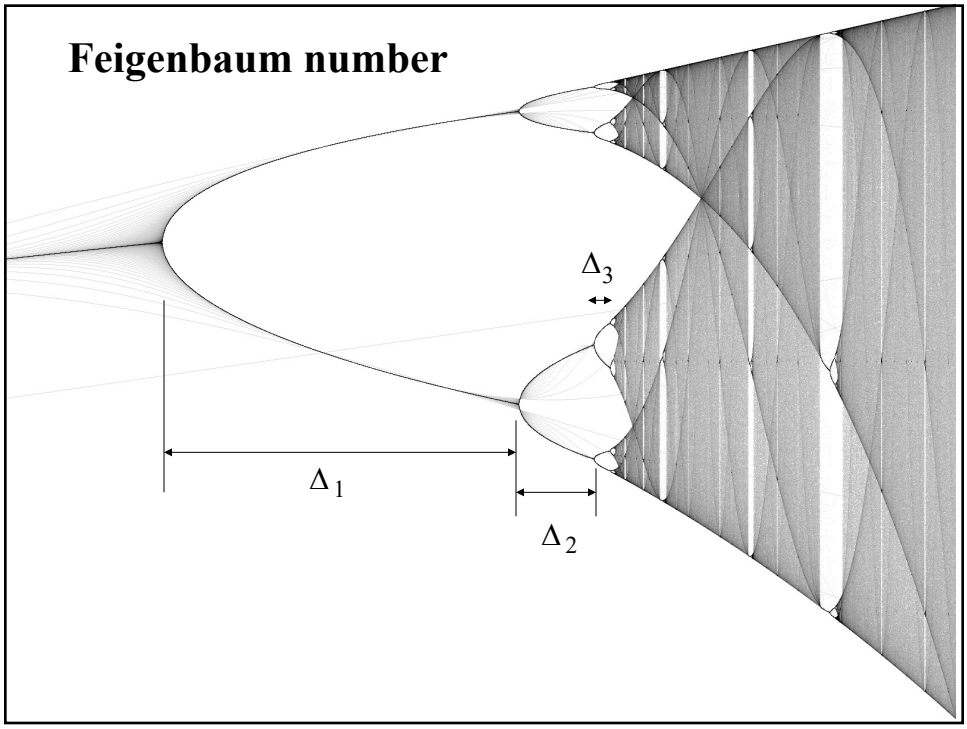


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**Universality!**

Feigenbaum number holds *for any 1D map with a quadratic maximum*.

Proof: renormalizations. See Strogatz § 10.7

*Don't take this too far, though…*

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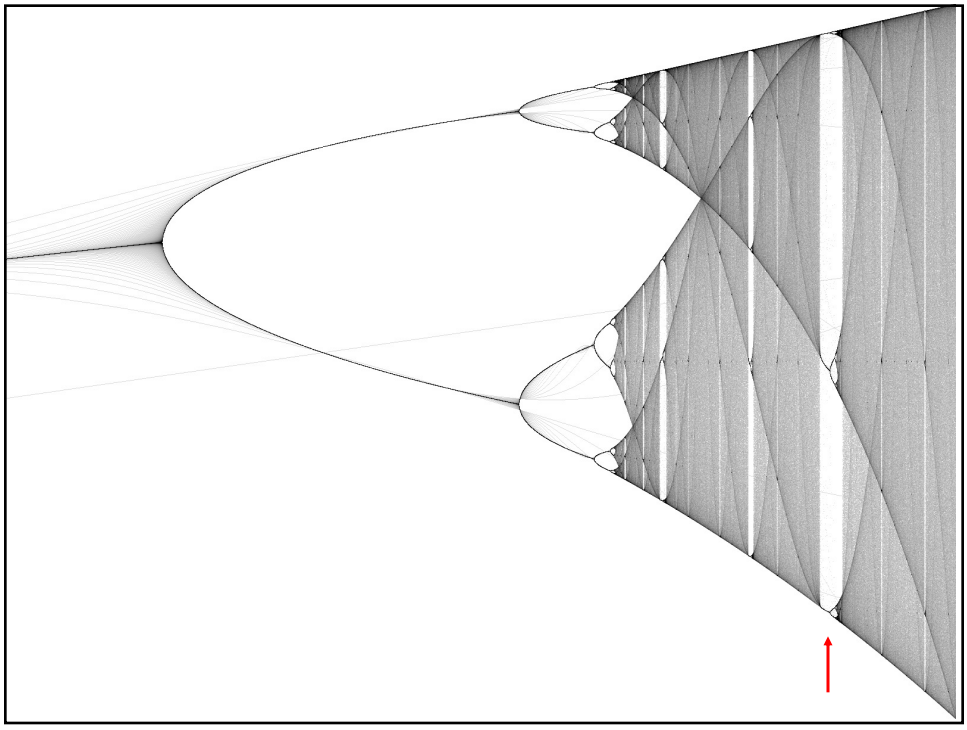

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- chaos
- veils/bands: places where chaotic attractor is dense (UPOs)
- period-doubling cascade @ low R
- *windows of order within the chaos, complete with their own period-doubling cascades (e.g., 3 to 6 to 12)*

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## There's something very special about 3…

- Sarkovskii (1964)

  3, 5, 7, …3x2, 5x2, …3x$2^2$, 5x$2^2$, … $2^2$, 2, 1

- Yorke (1975)

  **Period Three Implies Chaos**

  Tien-Yien Li; James A. Yorke

  *The American Mathematical Monthly*, Vol. 82, No. 10. (Dec., 1975), pp. 985-992.

- Metropolis *et al.* (1973)

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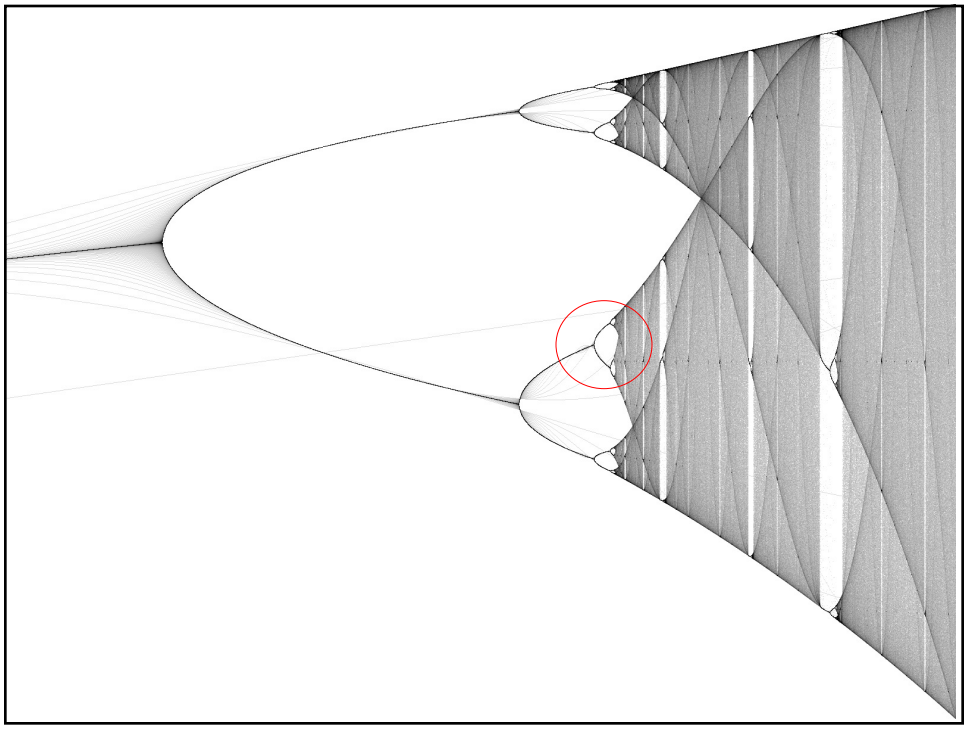

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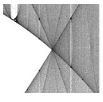
- chaos
- veils/bands: places where chaotic attractor is dense (UPOs)
- period-doubling cascade @ low R
- windows of order within the chaos, complete with their own period-doubling cascades (e.g., 3 to 6 to 12)
- *small copies of object embedded in it (fractal)*

(lots of other interesting stuff, too — e.g., Misiurewicz points)



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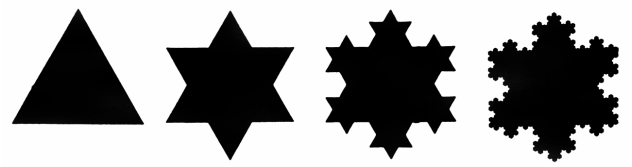
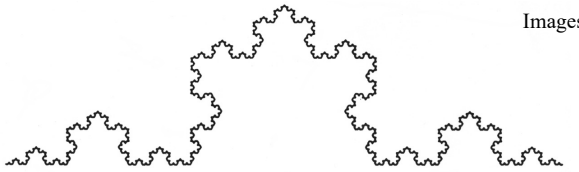
## Fractals

- non-integer Hausdorff dimension
- self-similar





Images from Gleick

Canonical example: the Cantor set!

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## Fractals in the wild

paulbourke.net/fractals/googleearth/

See also: coastlines, trees, lungs, clouds, snowflakes …

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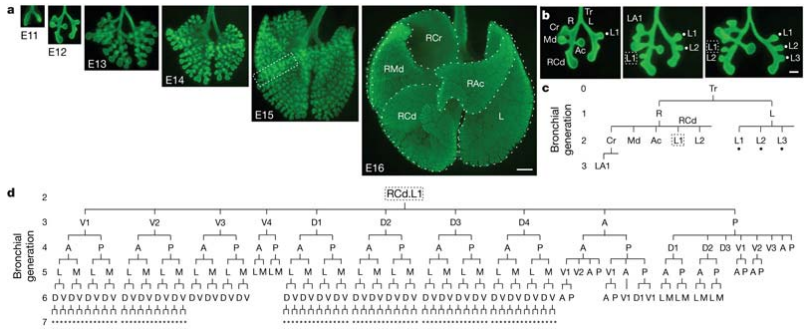
# ~23 generations of branches in the mammalian lung




**$2^{23}$ ~ 8 million branching events!**

**how does the epithelium fold 8 million times?**

Metzger et al, Nature, 2008

Slide courtesy of Celeste Nelson

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## Fractals and chaos…

The connection: *many (most)* chaotic systems have fractal state-space structure.

But **not** "*all.*"

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