So far: mostly about *maps.*

- discrete time systems:
  - time proceeds in clicks
  - “maps”
  - modeling tool: differ***ence*** equation

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Next up: *flows*

- continuous time systems:
  - time proceeds smoothly
  - “flows”
  - modeling tool: differ***ential*** equations

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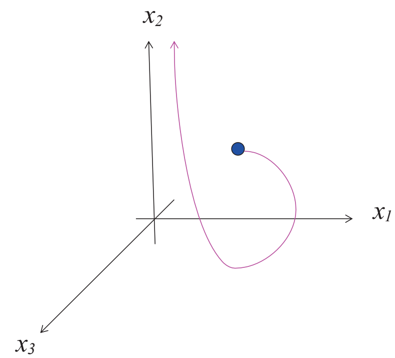
The basic representation in nonlinear dynamics: the state space



- State variables x_i
- Initial condition
- Trajectory

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The damped pendulum

- State variables?
- Initial condition?
- Trajectory?

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Fixed points of the simple pendulum

- Stable: perturbations shrink
- Unstable: perturbations grow

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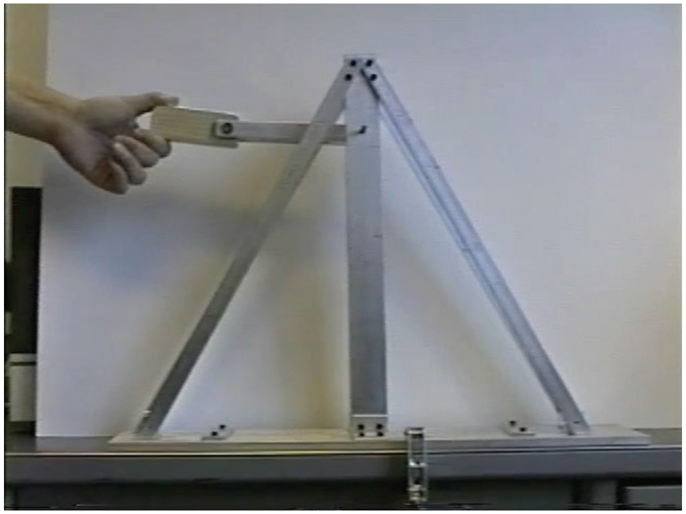

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Fixed points in the compound pendulum?

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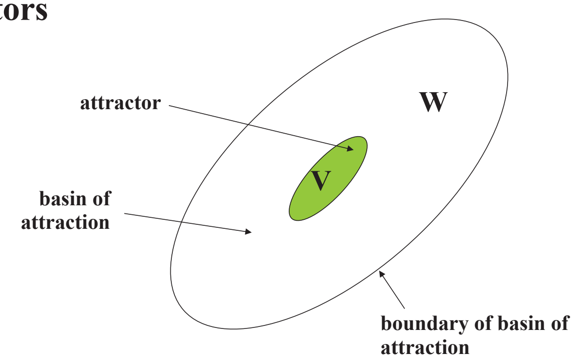
Attractors



- Attractors exist only in dissipative systems!
- Dissipation ⟺ contraction of state space under the influence of the dynamics
- Can still have chaos if no dissipation…just not chaotic *attractors*

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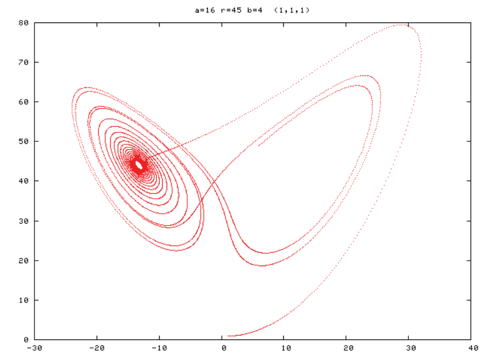
The archetype of chaos: Lorenz



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Deterministic Nonperiodic Flow[1]

EDWARD N. LORENZ

Massachusetts Institute of Technology

(Manuscript received 18 November 1962, in revised form 7 January 1963)

ABSTRACT

Finite systems of deterministic ordinary nonlinear differential equations may be designed to represent forced dissipative hydrodynamic flow. Solutions of these equations can be identified with trajectories in phase space. For those systems with bounded solutions, it is found that nonperiodic solutions are ordinarily unstable with respect to small modifications, so that slightly differing initial states can evolve into considerably different states. Systems with bounded solutions are shown to possess bounded numerical solutions.

A simple system representing cellular convection is solved numerically. All of the solutions are found to be unstable, and almost all of them are nonperiodic.

The feasibility of very-long-range weather prediction is examined in the light of these results.

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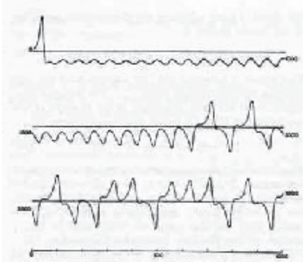

Fig. 1. Numerical solution of the convection equations. Graph of Y as a function of time for the first 1000 iterations (upper curve), second 1000 iterations (middle curve), and third 1000 iterations (lower curve).

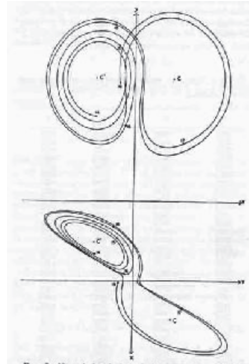

Fig. 2. Numerical solution of the convection equations. Projections on the X-Y-plane and the Y-Z-plane in phase space of the segment of the trajectory extending from iteration 1400 to iteration 1900. Numerals "14," "15," etc., denote positions at iterations 1400, 1500, etc. States of steady convection are denoted by C and C'.

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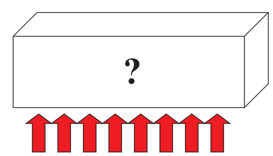
- Equations:

$x' = a(y - x)$

$y' = rx - y - xz$

$z' = xy - bz$



(first three terms of a Fourier expansion of the Navier-Stokes eqns)

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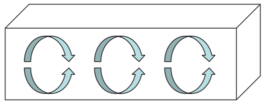

- State variables:
  - *x* convective intensity
  - *y* temperature
  - *z* deviation from linearity in the vertical convection profile

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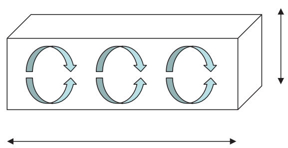
- Parameters:
  - *a* Prandtl number - fluids property
  - *r* Rayleigh number - related to ΔT 
  - *b* aspect ratio of the fluid sheet



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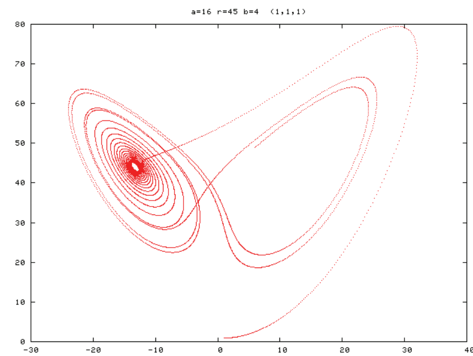

$x' = 16(y - x)$

$y' = 45x - y - xz$

$z' = xy - 4z$

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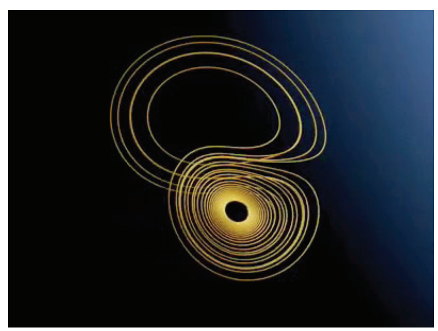

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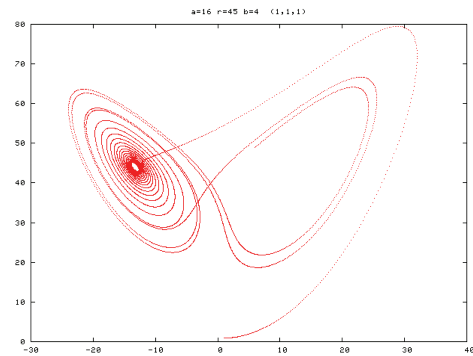

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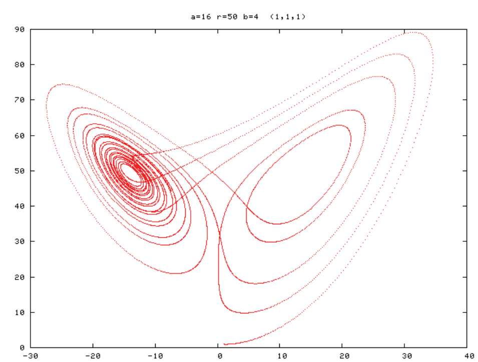
59

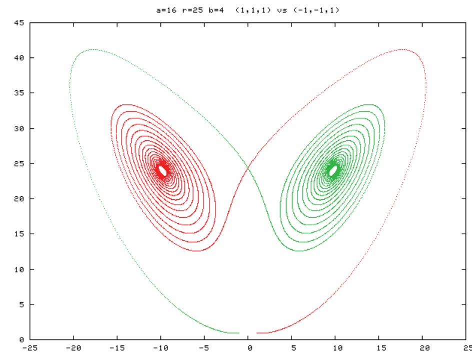


$x' = 16(y - x)$

$y' = \mathbf{50}x - y - xz$

$z' = xy - 4z$

60



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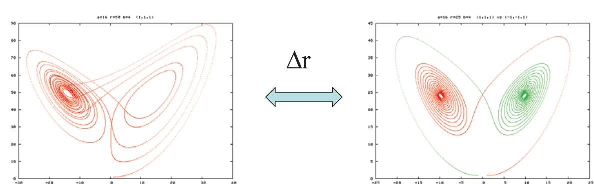
Recall: bifurcations

Qualitative changes in the dynamics—i.e., topological changes in the attractor—caused by changes in *parameters:*



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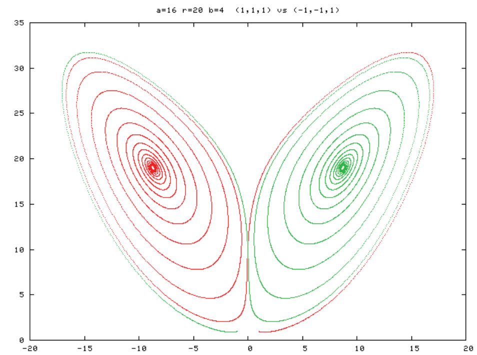

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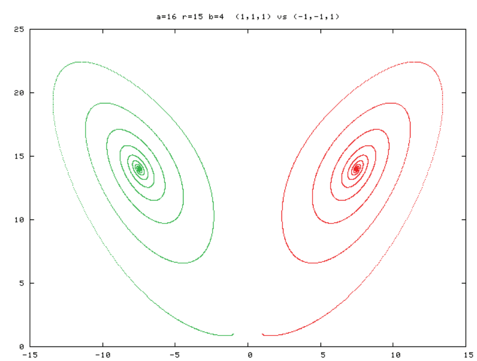

(NB: plain old *movement* of a fixed point isn't a bifurcation)

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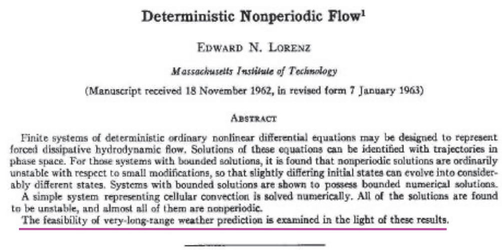
Before we leave Lorenz…



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The feasibility of very-long-range weather prediction is examined in the light of these results.

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Attractors

<u>Four types:</u>

- fixed points
- limit cycles (*aka* periodic orbits)
- quasiperiodic orbits
- chaotic attractors

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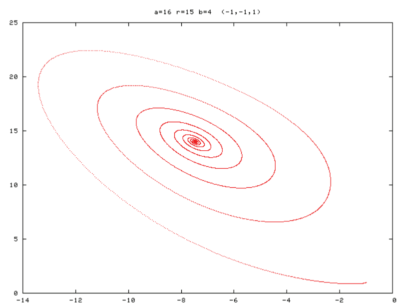
Attractors

- Fixed point



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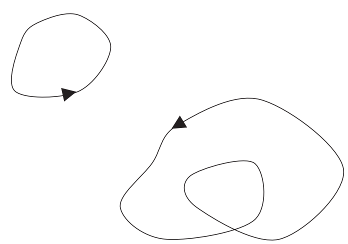
Attractors

- Limit cycle



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Attractors

- Quasi-periodic orbit…

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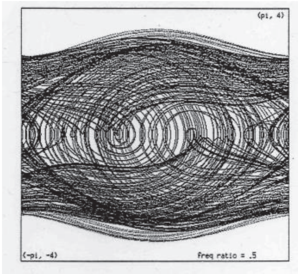
Chaotic (or “strange”) attractors:



State space


Real physical space

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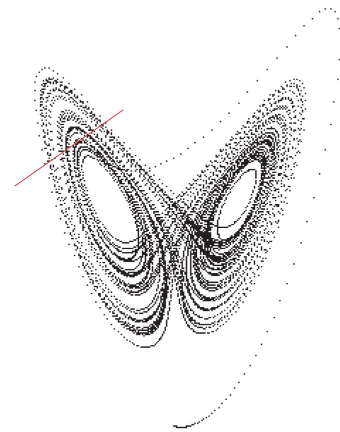
Chaotic attractors



• *often* fractal

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Fractals and chaos

The connection: *many (most)* chaotic systems have fractal state-space structure.

But **not** "*all*."

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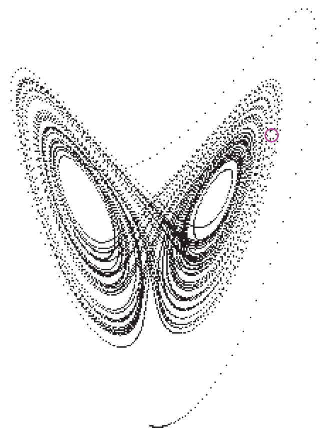
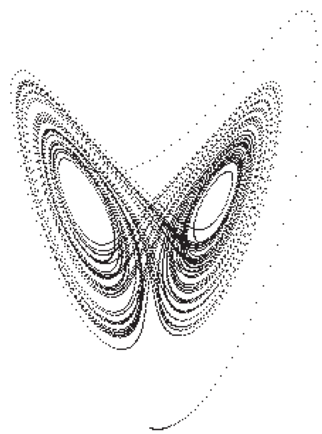
Chaotic attractors

- *often* fractal
- covered densely by trajectories



Chaotic attractors

- *often* fractal
- covered densely by trajectories
- SDOIC (exponential divergence of nearby trajectories)



Courtesy of Mike
Neuder

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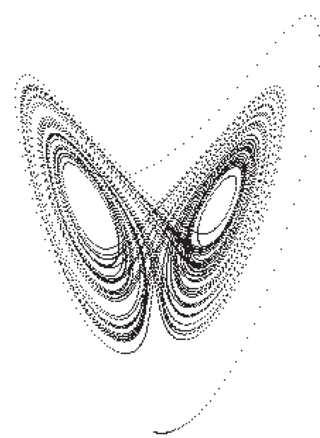
Chaotic attractors



- *often* fractal
- covered densely by trajectories
- SDOIC (**exponential** divergence of nearby trajectories)

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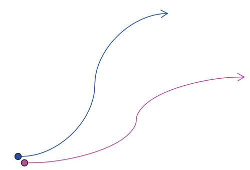
Lyapunov exponents and chaos

• distance between forward images of two nearby points grows as $e^{\lambda t}$ in the limit, as t → ∞



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Lyapunov exponents: some details

• negative λ_i compress state space; positive λ_i stretch it

• both can be going on at once, in different “directions"

• indeed, there are as many λ_i as there are state-space dimensions

• but the “directions” along which they act aren’t vectors, like they are in linear systems...

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Image from CGTrader

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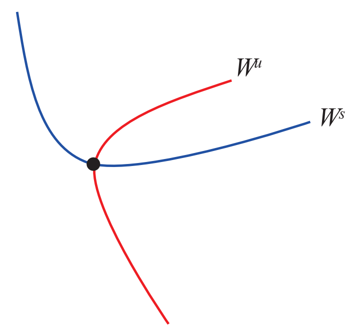
Lyapunov exponents: some details

- negative λ_i compress state space; positive λ_i stretch it
- both can be going on at once, in different "directions"
- indeed, there are as many λ_i as there are state-space dimensions
- but the "directions" along which they act aren't vectors.
- rather, the λ_i parametrize growth/shrinkage along the unstable and stable manifolds W^u and W^s

80

λ_i and the un/stable manifolds (W^u and W^s)



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Lyapunov exponents: some details

- negative λ_i compress state space; positive λ_i stretch it
- both can be going on at once, in different "directions"
- indeed, there are as many λ_i as there are state-space dimensions
- but the "directions" along which they act aren't vectors.
- rather, the λ_i parametrize growth/shrinkage along surfaces called unstable and stable manifolds W^u and W^s
- so you can think of λ_i as nonlinear analogs of eigenvalues s_i

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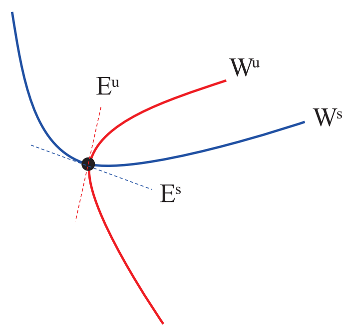
λ_i and the un/stable manifolds (W^u and W^s)



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Lyapunov exponents: more details

- long-term average in definition; biggest one (λ_1) dominates as t → ∞
- λ_i are about an *attractor*; they're the same for all initial conditions in one basin
- positive λ_1 is a signature of chaos
- but if all the λ_i were positive, there wouldn't be an attractor
- so there's a balance of expansion ($\lambda>0$) & compression ($\lambda<0$) going on in the state space of any dynamical system
- together with the geometry of the manifolds, this is what creates the structure of the attractor.

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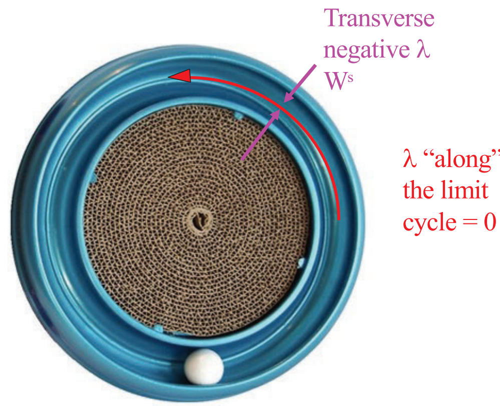
What that λ/manifold structure looks like for a periodic orbit:



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That negative transverse λ makes the periodic orbit stable in the face of perturbation:

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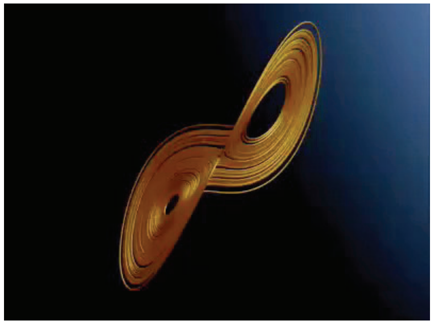
But there are also *unstable* periodic orbits…



There are an infinite number of these, of all periods, densely embedded in any chaotic attractor

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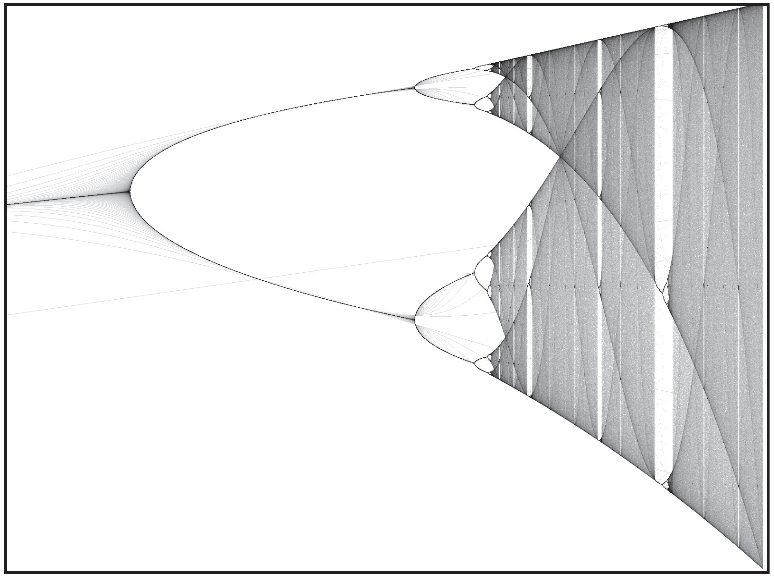

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Concepts: review

- State variable
- State space
- Initial condition
- Trajectory
- Attractor
- Basin of attraction

- Transient
- Fixed point (un/stable)
- Bifurcation
- Parameter
- Lyapunov exponent
- Stable & unstable manifolds

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Attractors

<u>Four types:</u>

- fixed points
- limit cycles (*aka* periodic orbits)
- quasiperiodic orbits
- chaotic attractors

A nonlinear system can have any number of attractors, of all types, sprinkled around its state space

Their basins of attraction (plus the basin boundaries) *partition* the state space

And there's no way, *a priori*, to know where they are, how many there are, what types, etc. (which is **not** the case in linear systems!)

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Conditions for chaos

(in continuous-time systems)

Necessary:

- Nonlinear
- At least three state-space dimensions

Necessary and sufficient:

- Cannot be solved in closed form (“nonintegrable,” in Hamiltonian parlance)

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